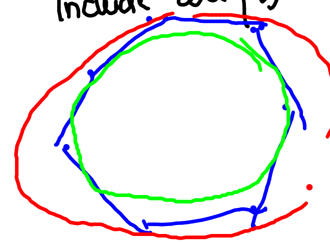
Convex optimization problem

Eg: Given a set of pts, find the tightest enclosing/enclosed ellipsoid.

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & \underbrace{f_i(x) \le b_i}_{x \in C}, \quad i = 1, \ldots, m
\end{array}
$$

Enclosing should include all pts



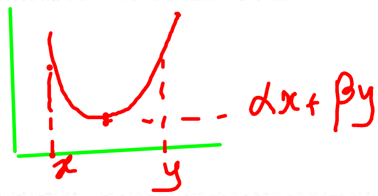
Enclosed ellipsoid should be included within the hull of these pts

- objective and constraint functions are convex:

  Initial part of course

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$



- includes least-squares problems and linear programs as special cases

Convex analysis: Calculus of inequalities

Convex geometry is easiest of geometries

Convex optimisation: Application of convex analysis

2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

$\theta_1 x_1 + \theta_2 x_2$

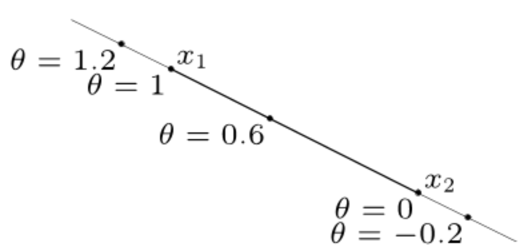
line through x_1, x_2: all points

[Affine] → ① $\theta_1 + \theta_2 = 1$

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$

② $\theta_1 \geq 0 \quad \theta_2 \geq 0$

③ $\theta_1 + \theta_2 = 1$ & $\theta_1, \theta_2 \geq 0$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$ → insight from linear algebra on geometry etc?

(conversely, every affine set can be expressed as solution set of system of linear equations)

2 Equivalent definitions of affine sets:

① $\forall\, x_1, x_2 \in S \qquad \theta_1 x_1 + \theta_2 x_2 \in S \qquad \theta_1 + \theta_2 = 1$

② $\{x \mid Ax = b\}$ for some $m \times n$ matrix A

Proof: ② $\Rightarrow$ ① is trivial since $Ax_1 = b$ & $Ax_2 = b$

$\Rightarrow A(\theta_1 x_1 + \theta_2 x_2) = b$ if $\theta_1 + \theta_2 = 1$

① $\Rightarrow$ ② ... suggestion: Subtract "p" $\in S$ from S

ie $S_p = S - p$ & show S_p is a v.s

For answer: pages 145 to 181 of

http://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

Rank of A

r=m=n	r=m<n	r=n<m
R=I	R=[I F]	$R=[I\ 0]^T$
Unique solution	Infinitely many solutions	0 or 1 solution

Figure 3.3: Summary of the properties of the solutions to the system of equations $Ax = b$.

$A_{m \times n}$ matrix

$A x_{particular} = b$

$A x_{nullspace} = 0$

Assign some values to free variables (x_2, x_4)

$$A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Example:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

(Gauss Elimination)

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \xrightarrow{E_{2,1}, E3,1} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\xrightarrow{E3,2} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

$r = 2 \quad m = 3 \quad n = 4$

Condition for solvability: $b_3 - b_1 - b_2 = 0$

Procedure to obtain A & b given an affine set S

① Let $p \in S$ $S - p$

Then claim: $\{x - p \mid x \in S\}$ is a vector space

Call it S_p

[H/W]

② Identify A s.t. $\forall\, x \in S_p,\ Ax = 0$

i.e rows of A could form basis of $S_p^{\perp}$

Basically $A\, x_{nullspace} = 0$

③ Identify $b = Ap$

Basically the $x_{particular}$ giving you $A\, x_{particular} = b$ in prev example

Thus:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable when $\mathbf{b}$ is in the column space $C(A)$.

Another way of describing solvability is:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable if a combination of the rows of A produces a zero row, the requirement on $\mathbf{b}$ is that the same combination of the components of $\mathbf{b}$ has to yield zero.

Steps to find $x_{particular}$:

1. $\mathbf{x}_{particular}$[2]: Set all free variables (corresponding to columns with no pivots) to 0. In the example above, we should set $x_2 = 0$ and $x_4 = 0$.
2. Solve $A\mathbf{x} = \mathbf{b}$ for pivot variables.

In this example:

$$x_1 + 2x_3 = b_1 \quad \& \quad 2x_3 = b_2 - 2b_1$$

$$\Rightarrow \quad \mathbf{x}_{particular} = \begin{bmatrix} b_2 + 3b_1 \\ 0 \\ \frac{b_2 - 2b_1}{2} \\ 0 \end{bmatrix}$$

Now: $\mathbf{x}_{complete} = \mathbf{x}_{particular} + \mathbf{x}_{nullspace}$ since $x_{nullspace}$ is s.t. $Ax_{nullspace} = 0$

$$\boxed{A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}}$$

Eg: if we choose $\mathbf{b} = [5\ 1\ 6]^T$, we get $\mathbf{x}_{particular} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$ &

$$x_{complete} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}}_{x_{particular}} + \underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{x_{nullspace}} \tag{3.36}$$

Show that $x_{complete} = \theta x_1 + (1-\theta) x_2$ for some $x_1, x_2 \in R^4$ & $\theta \in \mathbb{R}$

$\xRightarrow[\text{that}]{\text{Proves}}$ $\{x \mid Ax = b\}$ is an affine set

Q: What is a more generalised definition of affine sets?

More appropriate name when x_1 & x_2 are pts in real, finite dimensional Eucledian vector space $\mathbb{R}^n$ or $\mathbb{R}^{m \times n}$

Convex set

line segment between x_1 and x_2: all points

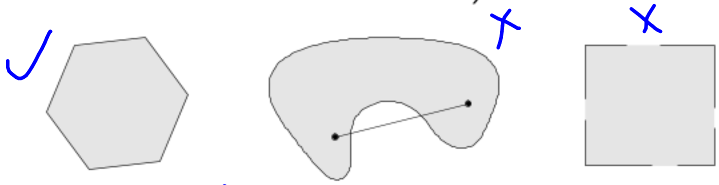
$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

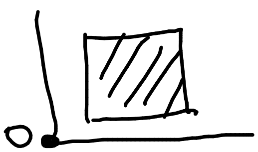
$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

examples (one convex, two nonconvex sets)



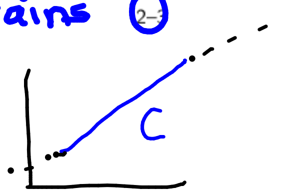
Aside: Convex set is connected: https://en.wikipedia.org/wiki/Connected_space

convex set can, but not necessarily contains '0'.



$\mathrm{aff}(C) \ni 0$

$0 \notin \mathrm{aff}(C)$



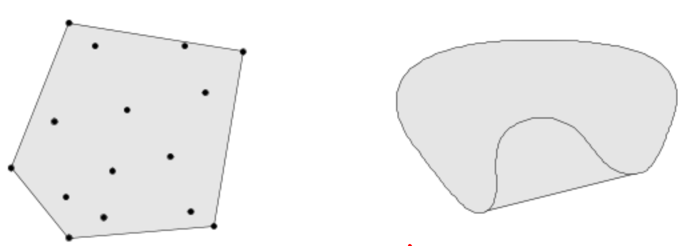
Convex combination and convex hull

convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k = \mathrm{conv}(\{x_1, x_2 .. x_k\})$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \ge 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S



conv(S) is always convex

Convex cone

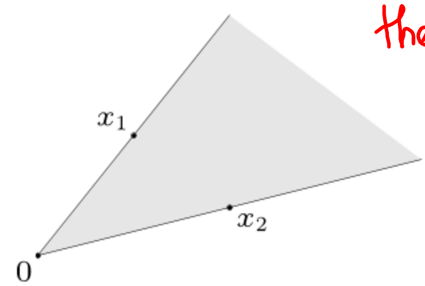
conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

If $\theta_1 = 0$ & $\theta_2 = 0$

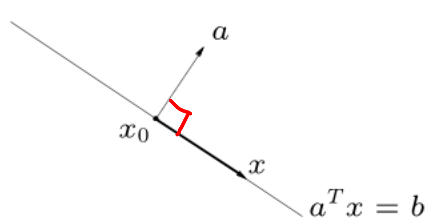
then $x = 0 \in$ Cone



convex cone: set that contains all conic combinations of points in the set

Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



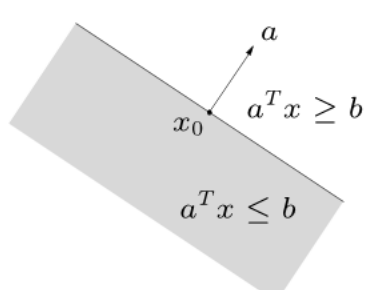
a is normal

$x_0 \in H$

$\{x \mid (x - x_0) \perp a\}$

$= \{x \mid x^T a = x_0^T a = b\}$

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

But NOT affine

Q: What is the relation between

A = affine set : $\theta_1 + \theta_2 = 1$

S = convex set : $\theta_1 + \theta_2 = 1 \quad \theta_1, \theta_2 \geq 0$

C = cone : $\theta_1, \theta_2 \geq 0$

Every affine set is convex
Every cone is convex.

- Family of affine sets is subset of family of convex sets
- Family of cones is subset of family of convex sets

Euclidean balls and ellipsoids

$\|x\|_2 = \sqrt{\sum x_i^2}$

(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

An ellipsoid is a Euclidean ball in a rotated space

ellipsoid: set of the form

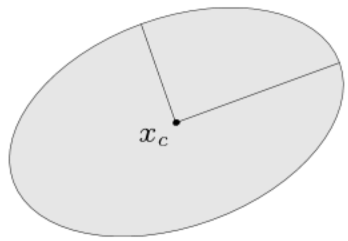
$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

$P > 0$ if all its eigenvalues are > 0

$P = U\Sigma U^T$

$\left((x-x_c)^T U\right)\Sigma^{-1}\left((x-x_c)^T U\right)^T$



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

verify: $A = (U\Sigma^{1/2})$

Q: Is P being p.d imp for convexity? For cone?

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

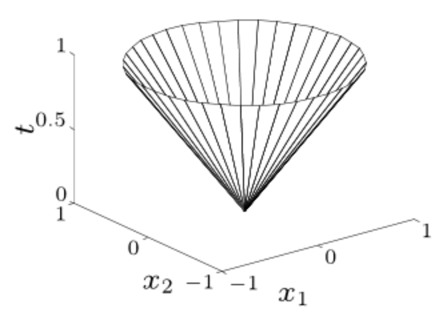
- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \le r\}$

norm cone: $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex