① LP is only a special case of CP with $K = \boxed{\mathbb{R}^+_n}$

② When will solution to CP = solution to CD?

CP: $\min_{x \in \mathbb{R}^n} \langle c, x \rangle$

s.t $Ax \geq_K b$

CD: $\max_{\lambda \in K^*} \langle b, \lambda \rangle$

s.t $A^* \lambda = c$

Since $K^{**} = K$, we saw that dual of CD is CP

While there exist multiple ways of writing CP & CD, hereafter we pick another standard format (to help you get used to various representations)

CP: $\min \langle c, x \rangle_V$

s.t $Ax = b$

$x \in K \subseteq V$

F_p $A: V \to \mathbb{R}^n$

CD: $\max \langle b, \lambda \rangle_{\mathbb{R}^n}$

s.t $c - A^* \lambda \in K^*$

$\lambda \in \mathbb{R}^n$

F_d $K^* \subseteq V$

STRONG DUALITY THM:

CP infeasible
CD feasible & int.

① Let CP or CD be infeasible & CD → ∞
let other be feasible & have an interior. Then the other is unbounded

→ CD is infeasible
CP is feasible & int
CP → −∞

② Let CP and CD be both feasible, and let one of them have an interior. Then there is 0 duality gap (solutions may not be attained)

③ Let CP and CD be both feasible and have interior. Then both have optimal solutions with 0 duality gap (solutions will be attained)

Theorems of alternatives (Farkas' lemmas)

① (Imagine instance of CP with $c=0$)

$\{x \mid \underbrace{Ax=b,\ x\in K}_{CP^*}\}$ $\quad$ $\{\lambda \mid \langle b,\lambda\rangle>0,\ \underbrace{-A^*\lambda\in K^*,\ \lambda\in\mathbb{R}^n}_{CD^*}\}$

& $\exists\,\hat{\lambda}$ s.t $-A^*\hat{\lambda}\in \operatorname{int}(K^*)$

One of them is non-empty iff the other is empty: Theorem of alternatives or Farkas lemma - - - -(Version I)

② (Imagine instance of CD, with $b=0$)

$\{x \mid \underbrace{Ax=0,\ x\in K,\ \langle c,x\rangle<0}_{CP^*}\}$ $\quad$ $\{\lambda \mid \underbrace{c-A^*\lambda\in K^*,\ \lambda\in\mathbb{R}^n}_{CD^*}\}$

& $\exists\,\hat{x}\in \operatorname{int}(K)$ s.t $A\hat{x}=0$

The parts highlighted in blue are required to prove that $C=\{y=Ax\in\mathbb{R}^m, x\in K\}$ and that $C'=\{t=s+A^*\lambda, \lambda\in\mathbb{R}^m, s\in K\}$ are closed convex sets respectively

Further, these were used to apply strict separating hyperplane theorem on $\{C,\{b\}\}$ and $\{C',\{c\}\}$ respectively

Now we apply theorem of alternatives (Farkas' lemma for conic inequalities) to prove the strong conic duality theorem

1. If $\mathcal{F}_d$ is empty and $\mathcal{F}_p$ is feasible & has an interior feasible solution, then we have $(\hat{x} \in \text{int}(K)$ & $\hat{\tau} = 1)$ form an interior feasible solution to

$$Ax - b\tau = 0 \quad (\hat{x}, \hat{\tau}) \in \text{int}(K \oplus R_{+})$$

We can now form an alternative system pair based on Farkas' lemma II

$$(*)\begin{cases} ① \left\{ (x,\tau) \mid Ax - b\tau = 0,\ \langle (c;-z), (x,\tau) \rangle < 0,\ (x,\tau) \in K \oplus R_{+} \right\} \\ ② \left\{ (\lambda, \theta) \mid \underbrace{c - A^{*}\lambda \in K^{*},\ -\langle b, \lambda \rangle}_{} + \theta = -z,\ \theta \in R_{+} \right\} \end{cases}$$

if $s = c - A^{*}\lambda$ then $(s, \theta) \in K^{*} \oplus R_{+}$

② is infeasible $\Rightarrow$ ① must have a feasible soln (x, τ). If at (x, τ) $\tau > 0$ then $\frac{1}{\tau} Ax = b$ & $x/\tau \in K$ & $\langle c, x/\tau \rangle < z$ for any z

$\Rightarrow$

Otherwise, $\tau = 0 \Rightarrow$ a new soln $\hat{x} + \alpha x$ is

$\Rightarrow$ Objective of CP is unbounded from below

(2) Let F_p be feasible & have an interior feasible soln & z be its infimum.

(A)

- We will make use of alternate system pair (*) from part (1) (since infimum is (z)).
- Now is infeasible $\Rightarrow$

from weak duality theorem

$\Rightarrow$

i.e we have soln (λ, s) s.t

$$A^* \lambda + s = c \qquad \langle b, \lambda \rangle = z \qquad s \in K^*$$

(B) SIMILARLY, one can prove that if z = supremum of dual (since CP & CD are both assumed to be feasible) and if F_D has non-empty interior then $\exists x \in K$ s.t $\langle c, x \rangle = z$

(3) If F_p & F_D are both feasible & both have interior, we can apply both parts (A) and (B) of above claim to get $\bar{x} \in F_p$ & $\bar{\lambda} \in F_D$ s.t $\inf CP = \langle \bar{x}, c \rangle = \langle \bar{\lambda}, b \rangle = \sup CD$

$\Rightarrow$ CP & CD have attainable optimal solns

Now we apply theorem of alternatives (Farkas' lemma for conic inequalities) to prove the strong conic duality theorem

1. If $\mathcal{F}_d$ is empty and $\mathcal{F}_p$ is feasible & has an interior feasible solution, then we have $\left(\hat{x} \in \text{int}(K) \;\&\; \hat{\tau} = 1\right)$ form an interior feasible solution to

$$Ax - b\tau = 0 \quad (\hat{x}, \hat{\tau}) \in \text{int}(K \oplus \mathbb{R}_{++})$$

We can now form an alternative system pair based on Farkas' lemma II, for any z

$$① \left\{ (x,\tau) \,\middle|\, Ax - b\tau = 0,\ \langle (c,-z),(x,\tau)\rangle < 0,\ (x,\tau) \in K \oplus \mathbb{R}_+ \right\}$$

Direct sum in vector spaces of closed convex cones is closed convex

$$\circledast \quad (C_1 \oplus C_2)^* = C_1^* \oplus C_2^*$$

$$② \left\{ (\lambda,\theta) \,\middle|\, \underbrace{c - A^*\lambda \in K^*,\ -\langle b,\lambda\rangle}_{} + \theta = -z,\ \theta \in \mathbb{R}_+ \right\}$$

if $s = c - A^*\lambda$ then $(s,\theta) \in K^* \oplus \mathbb{R}_+$

② is infeasible $\Rightarrow$ ① must have a feasible sol'n (x,τ). If at (x,τ) $\tau > 0$ then $\frac{1}{\tau}Ax = b$ & $x/\tau \in K$ & $\langle c, x/\tau\rangle < z$ for any z

$\Rightarrow \exists$ feasible sol'n in $\mathcal{F}_p$ whose objective $\to -\infty$

Otherwise, $\tau = 0 \Rightarrow$ a new soln $\hat{x} + \alpha x$ is feasible for the CP for any $\alpha > 0$ and as $\alpha \to \infty$ objective value of soln $\to -\infty$

$\Rightarrow$ Objective of CP is unbounded from below

2 Let F_p be feasible & have an interior feasible

A soln & Z be its infimum.

- We will make use of alternate system pair $*$ from part 1
- Now, for Z, 1 is infeasible (since infimum is 2) $\Rightarrow$ we have $\underbrace{\text{soln for 2}}_{(\lambda, \theta)}$

From weak duality theorem $\langle b, \lambda \rangle \leq Z$

$\Rightarrow \langle b, \lambda \rangle - Z = \theta \leq 0$ BUT $\theta \geq 0 \Rightarrow \theta = 0$

i.e. we have soln (λ, s) s.t

$$A^* \lambda + s = c \qquad \langle b, \lambda \rangle = Z \qquad s \in K^*$$

(that is, zero duality gap)

SIMILARLY, one can prove that if Z = supremum of

B dual (since CP & CD are both assumed to be feasible) and if F_D has non-empty interior then $\exists x \in K$ s.t $\langle c, x \rangle = Z$

3 If F_p & F_D are both feasible & both have interior, we can apply both parts A and B of above claim to get $\bar{x} \in F_p$ & $\bar{\lambda} \in F_D$ s.t $\inf CP = \langle \bar{x}, c \rangle = \langle \bar{\lambda}, b \rangle = \sup CD$

$\Rightarrow$ CP & CD have attainable optimal solns

Revisiting story of conic duality

$$\min_{\substack{Ax=b \\ x\in K\subseteq V \\ A:V\to\mathbb{R}^n}} \langle c,x\rangle_V = \min_{\substack{\lambda \leftarrow Ax=b \\ y \leftarrow x\in K\subseteq V \\ A:V\to\mathbb{R}^n}} \max_{\substack{\lambda\in\mathbb{R}^n \\ y\in K^*}} \langle c,x\rangle_V + \underbrace{\langle \lambda,(b-Ax)\rangle_{\mathbb{R}^n}}_{=0 \text{ at } \lambda^*, x^*} + \underbrace{\langle y,x\rangle_V}_{=0 \text{ at } y^*, x^*}$$

$$\geq \min_{x} \max_{\substack{\lambda\in\mathbb{R}^n \\ y\in K^*}} \langle c,x\rangle_V + \langle \lambda,(b-Ax)\rangle_{\mathbb{R}^n} + \langle y,x\rangle_V$$

$$\geq \max_{\substack{\lambda\in\mathbb{R}^n \\ y\in K^*}} \underbrace{\min_{x} \langle c,x\rangle_V + \langle \lambda,(b-Ax)\rangle_{\mathbb{R}^n} + \langle y,x\rangle_V}_{\min_x \langle c-A^*\lambda+y,x\rangle_V + \langle\lambda,b\rangle_{\mathbb{R}^n}}$$

$$\min_x \langle c-A^*\lambda+y,x\rangle_V + \langle\lambda,b\rangle_{\mathbb{R}^n} = \begin{cases} \langle\lambda,b\rangle \text{ if } A^*\lambda - c\in K^* \\ -\infty \text{ o/w}\end{cases}$$

$$= \max_{\substack{\lambda\in\mathbb{R}^n \\ A^*\lambda-c\in K^*}} \langle\lambda,b\rangle_{\mathbb{R}^n}$$

What more can you say if strong duality made these equalities?

$\langle \lambda^*,(b-Ax^*)\rangle = 0$

$\langle x^*, y^*\rangle = 0$

Q1: What if conditions in part ③ of strong duality hold? What more can we say? $\left(\langle\lambda^*,(b-Ax^*)\rangle=0,\ \langle x^*,y^*\rangle=0\right)$

Q2: What if we now have non-linear objective & non-linear constraints in the optimization problem

$$\boxed{\min_{x\in D} f(x) \ \text{s.t.}\ g_i(x)\leq 0 \text{ for } i=1..m}$$

$$\begin{cases} \min \ f(x) \\ \text{s.t } g_i(x) \le 0 \quad i = 1 \ldots m \\ h_j(x) = 0 \quad j = 1 \ldots k \end{cases}$$

We will generalize the inequalities & equalities

$$\min_x f(x) \quad \text{s.t } g_i(x) \le 0, \; h_j(x) = 0 \quad \geq \quad \min_x \max_{\lambda,\mu} \underbrace{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{k} \mu_j h_j(x)}_{L(x,\lambda,\mu)}$$

$$\text{s.t } g_i(x) \le 0, \quad h_j(x) = 0, \quad \lambda_i \ge 0 \quad \mu_j \in \mathbb{R}$$

$$\geq \min_x \max_{\lambda,\mu} L(x,\lambda,\mu) \qquad \lambda_i \ge 0, \; \mu_j \in \mathbb{R}$$

under strong duality

$\lambda_i^* g_i(x^*) = 0 \quad \forall i$

$\lambda_i^* \ge 0$

$\mu_j^* h_j(x^*) = 0 \quad \forall j$

$$\geq \max_{\lambda,\mu} \underbrace{\min_x L(x,\lambda,\mu)}_{L^*(\lambda,\mu) \text{ or lagrange dual fn.}} \qquad \lambda_i \ge 0, \; \mu_j \in \mathbb{R}$$

General weak duality result

$$= \underbrace{\max_{\lambda,\mu,\lambda \ge 0} L^*(\lambda,\mu)}_{\text{Dual opt problem}}$$

$$\min_{x \in \mathcal{D}} f(x)$$

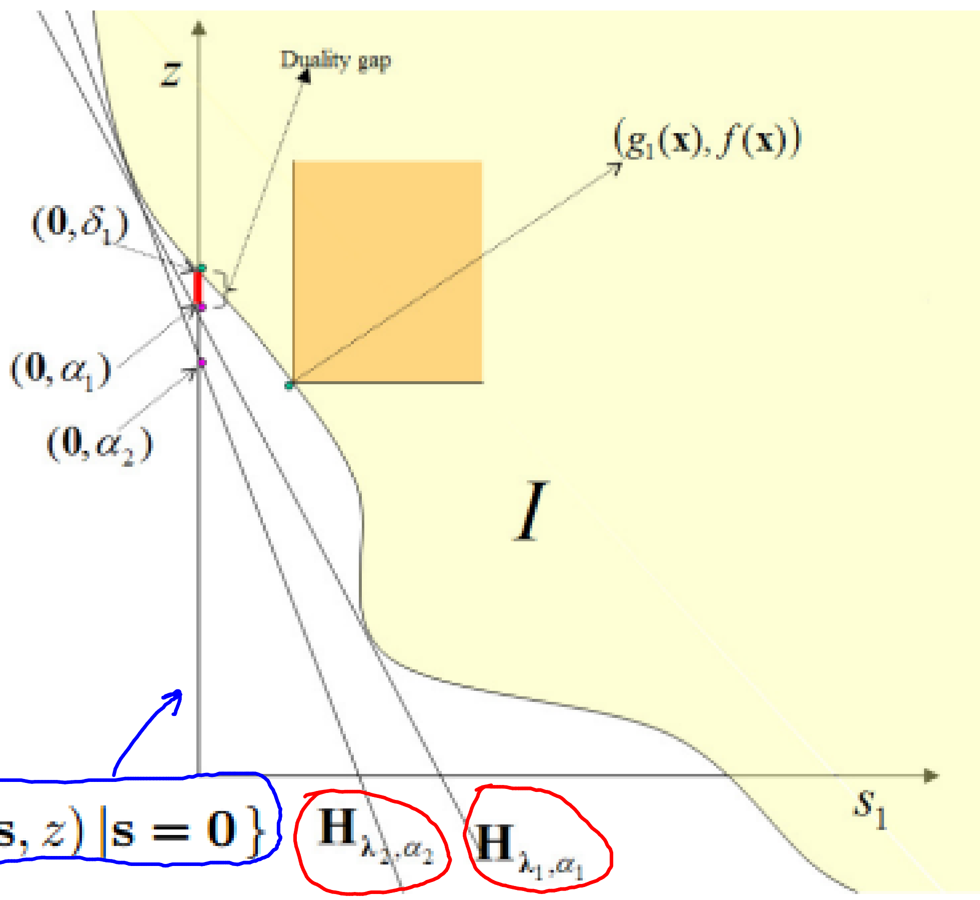
s.t. $g_i(x) \leq 0 \quad i = 1 \ldots m$

(The general dual problem & its geometric interpretation)

pg 292, sec 4.4.3 of http://www.cse.iitb.ac.in/~cs709/notes/BasicsOfConvexOptimization.pdf

Consider the set:

$$\mathcal{I} = \{(\mathbf{s}, z) \mid \mathbf{s} \in \Re^m,\ z \in \Re,\ \exists \mathbf{x} \in \mathcal{D} \text{ with } g_i(\mathbf{x}) \leq s_i\ \forall 1 \leq i \leq m,\ f(\mathbf{x}) \leq z\}$$



$$\mathcal{L} = \{(\mathbf{s}, z) \mid \mathbf{s} = \mathbf{0}\}$$

$$\mathcal{H}_{\lambda,\alpha} = \{(\mathbf{s}, z) \mid \lambda^T.\mathbf{s} + z = \alpha\}$$

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to} \quad & \mathcal{H}^{+}_{\lambda,\alpha} \supseteq \mathcal{I} \end{aligned}$$

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to} \quad & \lambda^T.\mathbf{s} + z \geq \alpha \ \forall (\mathbf{s}, z) \in \mathcal{I} \end{aligned}$$

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to} \quad & \lambda^T.\mathbf{s} + z \geq \alpha \ \forall (\mathbf{s}, z) \in \mathcal{I} \\ & \lambda \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \max \quad & \alpha \\ \text{subject to} \quad & \lambda^T.\mathbf{g}(\mathbf{x}) + f(\mathbf{x}) \geq \alpha \ \forall \mathbf{x} \in \mathcal{D} \\ & \lambda \geq \mathbf{0} \end{aligned}$$

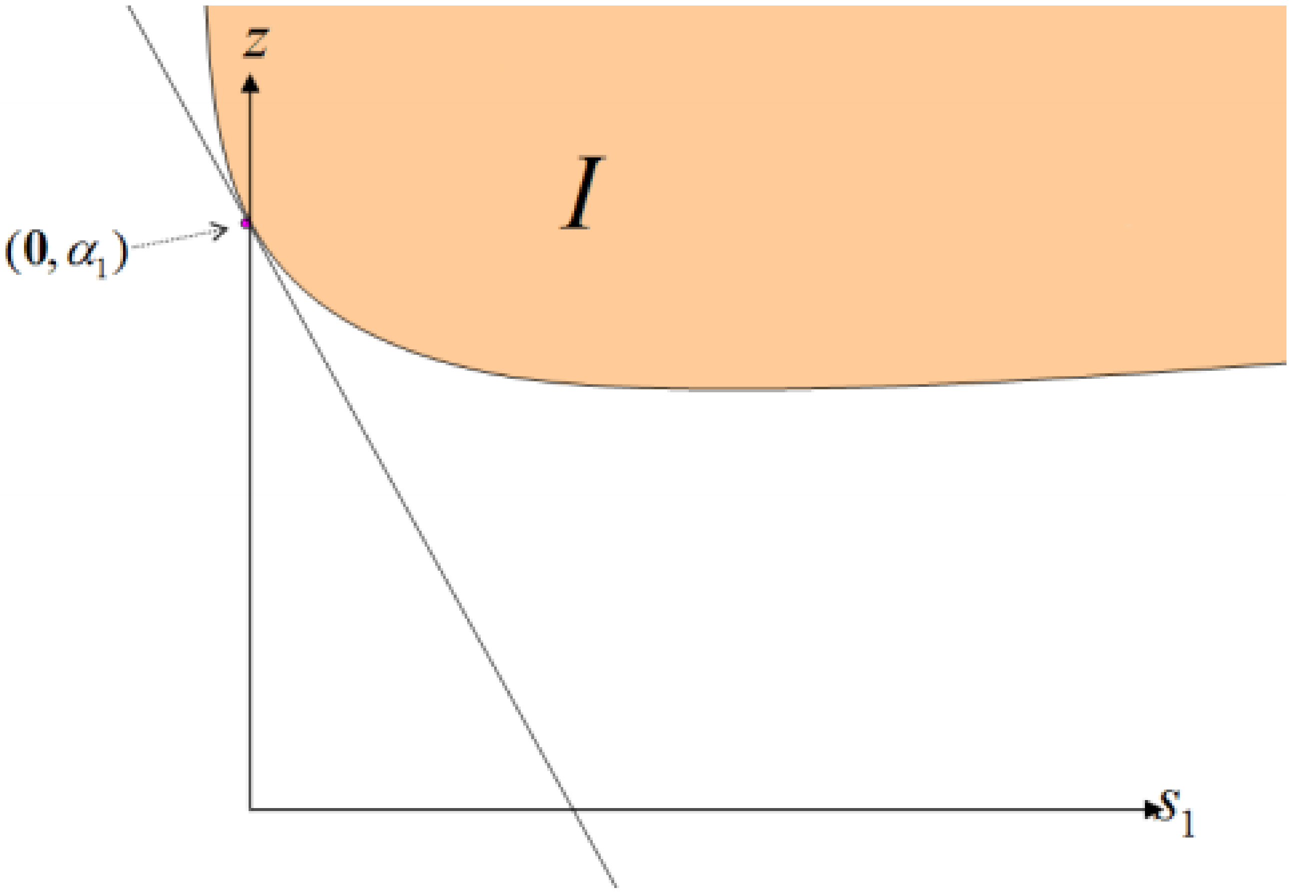
$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & L(\mathbf{x}, \lambda) \geq \alpha \ \forall \mathbf{x} \in \mathcal{D} \\
& \lambda \geq \mathbf{0}
\end{array}
$$

Since, $L^*(\lambda) = \min\limits_{\mathbf{x} \in \mathcal{D}} L(\mathbf{x}, \lambda)$, we can deal with the equivalent

$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & L^*(\lambda) \geq \alpha \\
& \lambda \geq \mathbf{0}
\end{array}
$$

This problem can be restated as

$$
\begin{array}{ll}
\max & L^*(\lambda) \\
\text{subject to} & \lambda \geq \mathbf{0}
\end{array}
$$


z
I
$(\mathbf{0}, \alpha_1)$
s_1