Revisiting story of conic duality

$$\min_{\substack{Ax=b\\ x\in K\subseteq V}} \langle c,x\rangle_V \;=\; \min_{\substack{Ax=b\\ x\in K\subseteq V\\ A:V\to\mathbb{R}^n}} \; \max_{\substack{\lambda\in\mathbb{R}^n\\ y\in K^*}} \; \langle c,x\rangle_V + \underbrace{\langle \lambda,(b-Ax)\rangle_{\mathbb{R}^n}}_{=0 \text{ at } \lambda^*,x^*} + \underbrace{\langle y,x\rangle_V}_{=0 \text{ at } y^*,x^*}$$

$A:V\to\mathbb{R}^n$

$\lambda \leftarrow Ax=b$, $\quad y \leftarrow x\in K\subseteq V$

$$\geq \min_{x} \; \max_{\substack{\lambda\in\mathbb{R}^n\\ y\in K^*}} \; \langle c,x\rangle_V + \langle \lambda,(b-Ax)\rangle_{\mathbb{R}^n} + \langle y,x\rangle_V$$

$$\geq \max_{\substack{\lambda\in\mathbb{R}^n\\ y\in K^*}} \; \underbrace{\min_{x} \; \langle c,x\rangle_V + \boxed{\langle \lambda,(b-Ax)\rangle_{\mathbb{R}^n} + \langle y,x\rangle_V}}_{\min_x \langle c-A^*\lambda+y,\,x\rangle_V + \langle\lambda,b\rangle_{\mathbb{R}^n}}$$

$$\min_x \langle c-A^*\lambda+y,\,x\rangle_V + \langle\lambda,b\rangle_{\mathbb{R}^n} = \begin{cases} \langle\lambda,b\rangle & \text{if } A^*\lambda - c\in K^* \\ -\infty & \text{o/w} \end{cases}$$

$$= \max_{\substack{\lambda\in\mathbb{R}^n\\ A^*\lambda-c\in K^*}} \langle \lambda,b\rangle_{\mathbb{R}^n}$$

What more can you say if strong duality made these equalities?

$\langle \lambda^*,(b-Ax^*)\rangle = 0$

$\langle x^*,y^*\rangle = 0$

Q1: What if conditions in part (3) of strong duality hold? What more can we say? $\left(\langle\lambda^*,(b-Ax^*)\rangle=0,\ \langle x^*,y^*\rangle=0\right)$

Q2: What if we now have non-linear objective & non-linear constraints in the optimization problem

$$\boxed{\min_{x\in D} f(x) \quad \text{s.t. } g_i(x)\leq 0 \text{ for } i=1..m}$$

$$\begin{cases} \min \; f(x) \\ \text{s.t } g_i(x) \le 0 \quad i = 1 \dots m \\ \quad h_j(x) = 0 \quad j = 1 \dots k \end{cases}$$

We will generalize the inequalities & equalities

$$\min_x f(x) \quad \text{s.t } g_i(x) \le 0, \; h_j(x) = 0 \;\; \geq \;\; \min_x \max_{\lambda,\mu} \underbrace{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{k} \mu_j h_j(x)}_{L(x,\lambda,\mu)}$$

$$\text{s.t } g_i(x) \le 0, \quad h_j(x) = 0, \quad \lambda_i \geq 0, \quad \mu_j \in \mathbb{R}$$

$$\geq \min_x \max_{\lambda,\mu} L(x,\lambda,\mu), \quad \lambda_i \geq 0, \; \mu_j \in \mathbb{R}$$

under strong duality

$$\lambda_i^* g_i(x^*) = 0 \;\; \forall i$$

$$\lambda_i^* \geq 0$$

$$\mu_j^* h_j(x^*) = 0 \;\; \forall j$$

$$\geq \max_{\lambda,\mu} \underbrace{\min_x L(x,\lambda,\mu)}_{L^*(\lambda,\mu)}, \quad \lambda_i \geq 0, \; \mu_j \in \mathbb{R}$$

$L^*(\lambda,\mu)$ or lagrange dual fn.

General weak duality result

$$= \underbrace{\max_{\lambda,\mu,\lambda \geq 0} L^*(\lambda,\mu)}_{\text{Dual opt problem}}$$

$$\min_{x \in D} f(x)$$

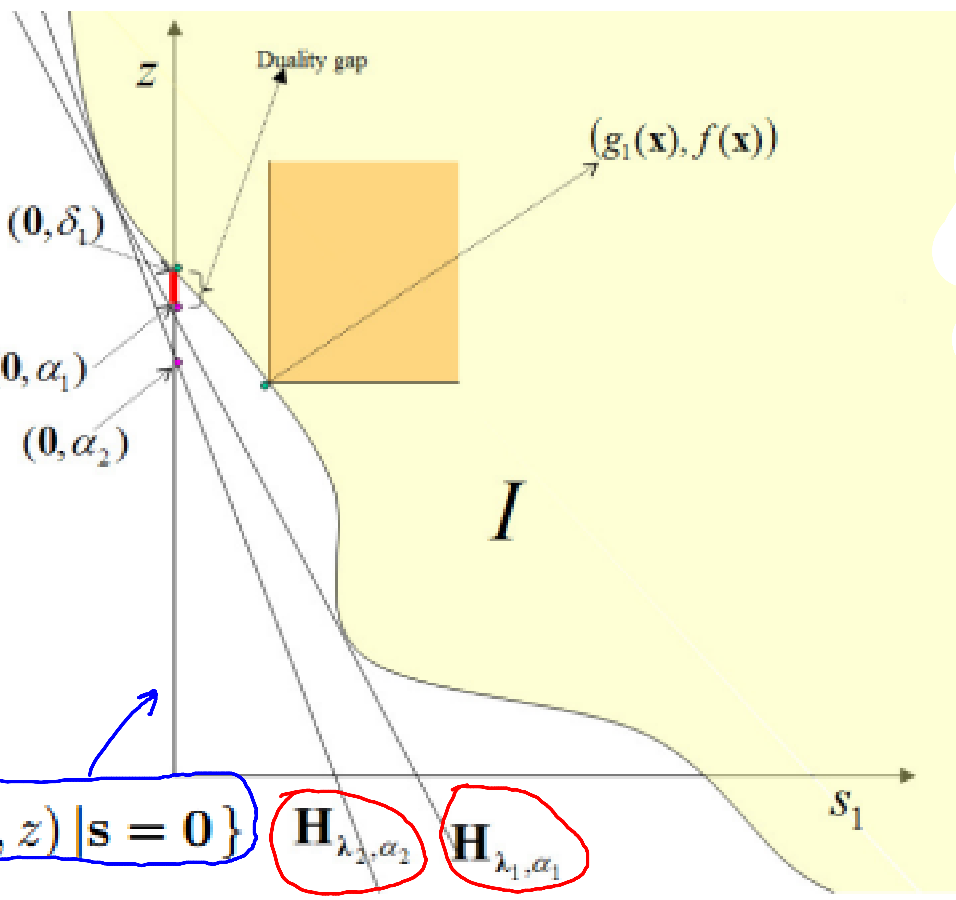
s.t. $g_i(x) \leq 0 \quad i = 1 \ldots m$

(The general dual problem & its geometric interpreta-tion)

pg 292, sec 4.4.3 of http://www.cse.iitb.ac.in/~cs709/notes/BasicsOfConvexOptimization.pdf

Consider the set:

$$\mathcal{I} = \{(\mathbf{s}, z) \mid \mathbf{s} \in \Re^m,\ z \in \Re,\ \exists \mathbf{x} \in \mathcal{D} \ with\ g_i(\mathbf{x}) \leq s_i\ \forall 1 \leq i \leq m,\ f(\mathbf{x}) \leq z\}$$



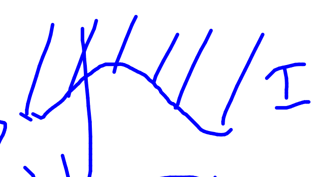
Smallest z value in $\mathcal{I}$ for $s_1 \leq 0$ will be for $s_1 = 0$

Since $(s_1, z_1) \in \mathcal{I}$ $\Rightarrow (s_2, z_2) \in \mathcal{I}$ $\forall s_2 \geq s_1$ & $z_2 \geq z_1$

$$\mathcal{L} = \{(\mathbf{s}, z) \mid \mathbf{s} = \mathbf{0}\}$$

$$\mathcal{H}_{\lambda,\alpha} = \{(\mathbf{s}, z) \mid \lambda^T.\mathbf{s} + z = \alpha\}$$

Thus $\Rightarrow$ is not possible!



$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & \mathcal{H}^{+}_{\lambda,\alpha} \supseteq \mathcal{I}
\end{array}
$$

$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & \lambda^T.\mathbf{s} + z \geq \alpha \ \forall (\mathbf{s}, z) \in \mathcal{I}
\end{array}
$$

$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & \lambda^T.\mathbf{s} + z \geq \alpha \ \forall (\mathbf{s}, z) \in \mathcal{I} \\
& \lambda \geq \mathbf{0}
\end{array}
$$

$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & \lambda^T.\mathbf{g}(\mathbf{x}) + f(\mathbf{x}) \geq \alpha \ \forall \mathbf{x} \in \mathcal{D} \\
& \lambda \geq \mathbf{0}
\end{array}
$$