Convex cone

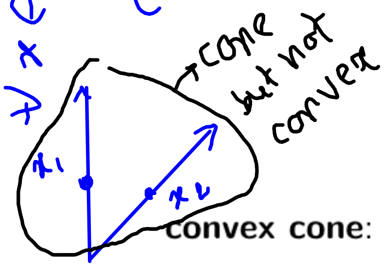
Cone: C is a cone if $\forall x \in C$, $\theta x \in C$ for $\theta \geq 0$

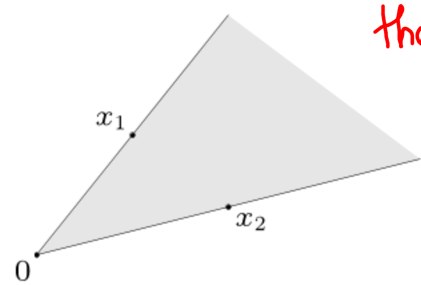
conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

If $\theta_1 = 0$ & $\theta_2 = 0$ then $x = 0 \in$ Convex Cone

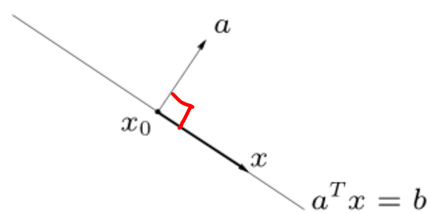




convex cone: set that contains all conic combinations of points in the set

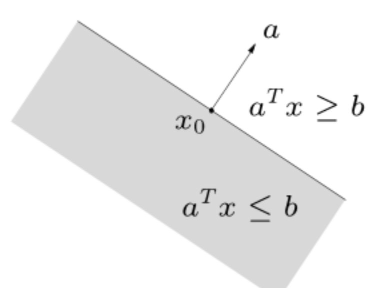
Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



a is normal
$x_0 \in H$
$\{x \mid (x - x_0) \perp a\}$
$= \{x \mid x^T a = x_0^T a = b\}$

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

But NOT affine

Q: What is the relation between

A = affine set : $\theta_1 + \theta_2 = 1$

S = convex set : $\theta_1 + \theta_2 = 1 \quad \theta_1, \theta_2 \geq 0$

C = convex cone: $\theta_1, \theta_2 \geq 0$

Every affine set is convex

Every convex cone is convex.

- Family of affine sets is subset of family of convex sets
- Family of cones is subset of family of convex sets

Thus: (a) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = set of all convex combinations of pts in S

(b) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = smallest convex set that contains S [Prove as h/w]

Also: The idea of a convex combination can be generalised to include infinite sums, integrals, and, in the most general form, probability distributions

Similarly: (a) $\underbrace{\text{Conic/Affine hull}(S)}_{\text{conic}(S) \text{ or aff}(S)}$ = set of all conic/affine combination of pts in S

(b) $\underbrace{\text{Conic/Affine hull}(S)}_{\text{conic}(S) \text{ or aff}(S)}$ = smallest conic/affine set that contains S

Euclidean balls and ellipsoids

$\|x\|_2 : \sqrt{\sum x_i^2}$

(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

An ellipsoid is a Euclidean ball in a rotated space

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

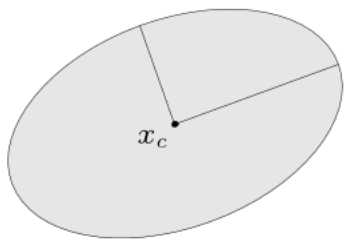
As per this defn P being p.d is necessary

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

$P > 0$ if all its eigenvalues are > 0

$P = U \Sigma U^T$

$\left((x-x_c)^T U\right) \Sigma^{-1} \left((x-x_c)^T U\right)^T$



Write down relation between A & P

other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Scaling & rotation

Verify: $A = (U \Sigma^{1/2})$

Q: Is P being p.d necessary convexity? For cone!

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$ (triangle inequality)

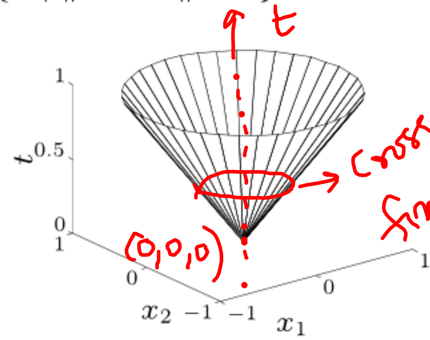
notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \le r\}$

pointedness

norm cone: $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone



Cross section fixed t has

Giving you a cross section as norm ball in $\mathbb{R}^2$

norm balls and cones are convex

If $A \succ 0$ (positive definite) ① & ③ dont assume Symmetry

① $\text{real}(\lambda) > 0$ Eg: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $x^T A x = x_1^2 + x_2^2$

② If $A \in S^n$ (S^n is space of all symmetric matrices) then we can show that all its eigenvalues are real (H/W)

③ $x^T A x > 0 \ \forall x \in \mathbb{R}^n, x \neq 0$ & $x^T A x = 0$ iff $x = 0$

Assumes $A \in S^n$

④ $x^T A y$ is an inner product (by virtue of defn of inner prod)

⑤ $A = LL^T$ L is lower triangular &

Assumes A is symmetric

$A = Q \Sigma Q^T$ where Q is orthonormal & Σ is positive diagonal matrix

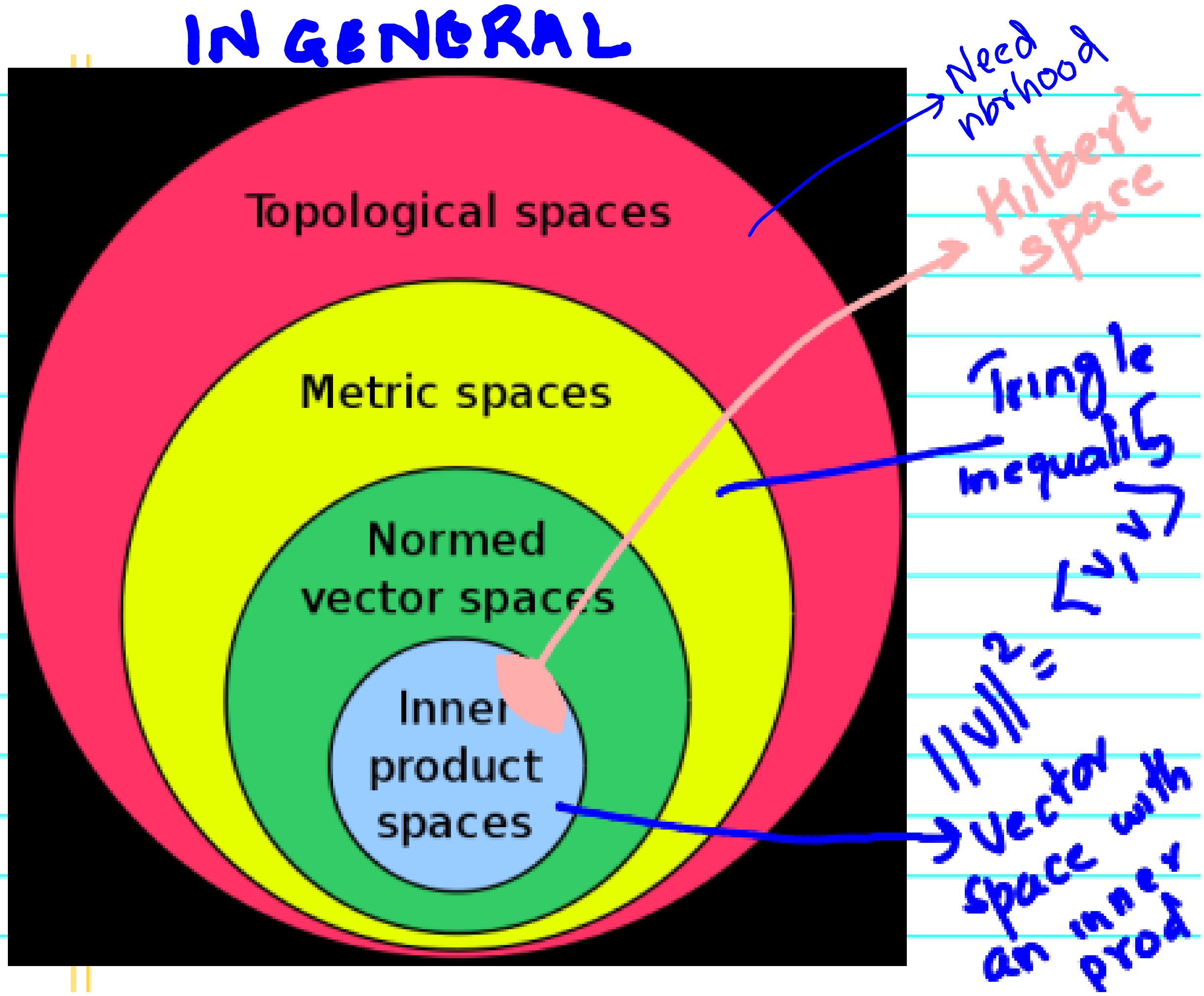
⑥ $$A \doteq \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{anti-symmetric}}$$

$$x^T A x = \underbrace{\frac{1}{2} x^T (A + A^T) x} + \underbrace{\frac{1}{2} x^T (A - A^T) x}_{0}$$

It does not hurt in convex analysis to consider only symmetric part of A ie to assume A is symmetric

$$x^T A x = (x^T A x)^T = x^T A^T x$$

IN GENERAL



Source: http://en.wikipedia.org/wiki/Space_(mathematics)

A hierarchy of mathematical spaces: The inner product induces a norm. The norm induces a metric. The metric induces a topology.

Topological space: Set of points along with a set of neighborhoods of each point, with certain axioms required to be satisfied by the pts & their nbrhoods

Metric space: Set of points with a notion of "distance" between elements
$d(x,y)$ must be ⓐ non-negative ⓑ $d(x,y)=0$ iff $x=y$ ⓒ symmetric ⓓ satisfy triangle inequality

Assuming you have understood vector space

Normed vector space: A vector space on which a norm is defined. (See page number 4 for definition of norm)

Definitions: In topological space, $\{x_i\}$ could converge to a limit $\lim_{i\to\infty} x_i$

- $\lim_{i\to\infty} \frac{1}{i} = 0$

//en.wikipedia.org/wiki/Limit_point

$cl(S)$ when S is a topological space

→ Should consist of S

union with

→ Should consist of $\lim_{i\to\infty} x_i$ for every convergent sequence $\{x_i\} \subseteq S$

For general topological space

S is closed if $cl(S) = S$

S is open if S^c is closed

$\text{interior}(S) = \bigcup_{\substack{S' = \text{open} \\ S' \subseteq S}} S'$

$bnd(S) = cl(S) - int(S)$

$\stackrel{?}{=} cl(S) \cap cl(S^c)$

with norm $\|\cdot\|$

$\forall x \in S, \exists \epsilon > 0$ s.t. $\{y \mid \|y - x\| \leq \epsilon\} \subseteq S$ — open set in Normed v.s

$bnd(S) = \partial(S)$

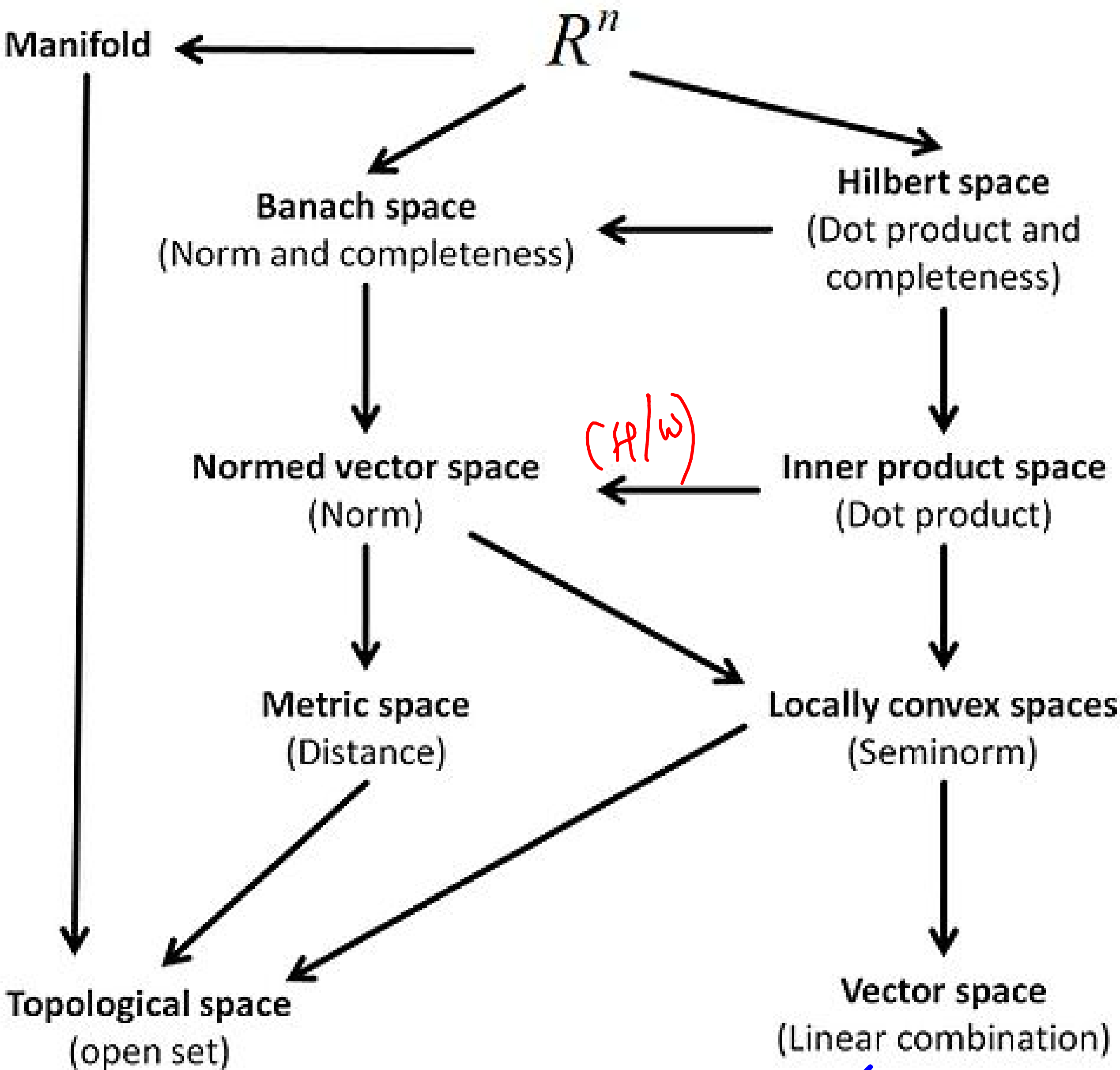
x belongs to the normed space

$cl(S) = \{x \mid \forall \epsilon > 0 \; S \cap \{y \mid \|x - y\| < \epsilon\} \neq \phi\}$

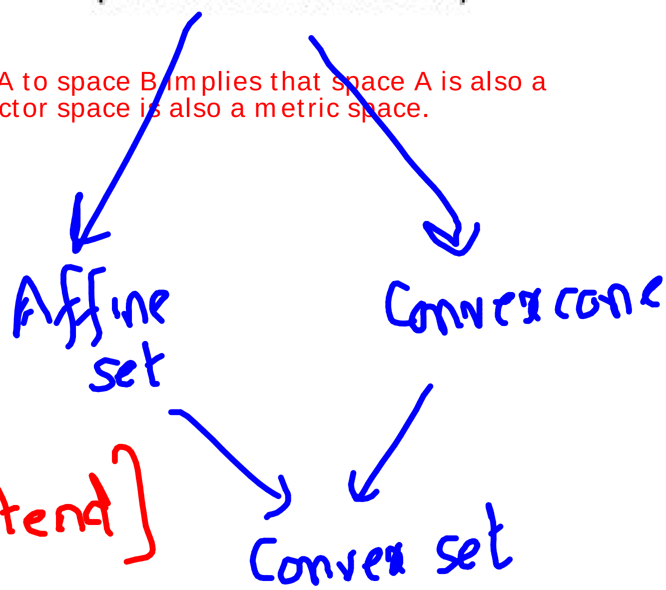
$int(S) = \{x \mid x \in S \text{ s.t. } \exists \epsilon > 0 \text{ s.t. } \{y \mid \|x - y\| < \epsilon\} \subseteq S\}$

$relbnd(S) = cl(S) - relint(S)$

$relint(S) = \{x \mid x \in S \text{ s.t. } \exists \epsilon > 0 \text{ s.t. } \{y \mid \|x - y\| < \epsilon\} \cap aff(S) \subseteq S\}$



Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



[H/W: Confirm
or contend]

[H/w: Prove that "normed" space is a "metric" space]

Inner product space: It is a vector space over a field of scalars along with an inner product

eg: $\mathbb{R}$

an algebraic structure with addition, subtraction, multiplication & division

associative & commutative

must be commutative, associative & distributive

multiplicative inverse must exist

ⓐ (Conjugate) symmetry:

$$\langle x, y\rangle = \langle y, x\rangle$$

ⓑ Linearity in the first argument

$$\langle ax, y\rangle = a\langle x, y\rangle$$

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

ⓒ Positive definiteness:

$\langle x, x\rangle \geq 0$ with equality iff $x = 0$