Prove that the real part of the eigenvalue of a (not necessarily symmetric) positive definite matrix is always positive.

Let A be positive definite, $A \in \mathbb{R}^{n \times n}$

$: \forall x \in \mathbb{R}^n, x \neq 0 \Rightarrow x^T A x > 0$

Defn

Note: $A = \begin{bmatrix} a & a \\ -a & a \end{bmatrix}, a > 0$

is p.d $(a x_1^2 + a x_2^2 > 0)$

and its eigenvalues are

$a + ai$ & $a - ai$

Let $\lambda = \lambda_1 + i\lambda_2$ be any eigenvalue of A

$v = v_1 + i v_2$ be corresponding eigenvector

$$Av = \lambda v \Rightarrow (A - \lambda_1 I - i\lambda_2 I)(v_1 + i v_2) = 0$$

$$\Rightarrow (A - \lambda_1 I) v_1 + \lambda_2 I v_2 = 0 \rightarrow \text{premultiply by } v_1^T$$

$$(A - \lambda_1 I) v_2 - \lambda_2 I v_1 = 0 \rightarrow \text{premultiply by } v_2^T$$

$+$

$$v_1^T (A - \lambda_1 I) v_1 + v_2^T (A - \lambda_1 I) v_2 = 0$$

$$\Rightarrow \lambda_1 = \frac{v_1^T A v_1 + v_2^T A v_2}{v_1^T v_1 + v_2^T v_2} > 0$$

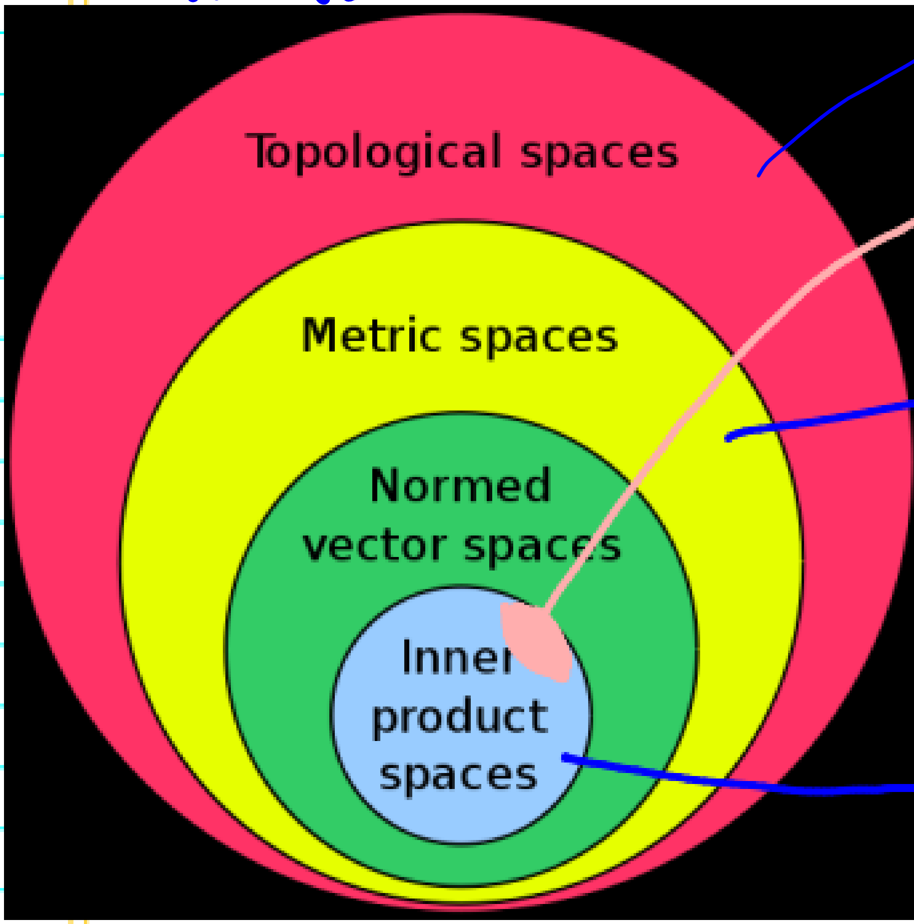
Q: How would your defn of positive definiteness be different if you allowed $A \in \mathbb{C}^{n \times n}$ & $x \in \mathbb{C}^n$?

$\mathbb{C}$ = complex nos

Do same steps hold?

$$Re(x^* A x) > 0 \quad ? \quad \equiv \quad Re(x^* A^* x) > 0$$

IN GENERAL



Need nbrhood {$x \in N(x)$, intersection of nbrhoods is nbrhood..}

4 axioms

H/W: Determine what makes nbrhood open

Hilbert space

Tringle inequality

$\|v\|^2 = \langle v, v \rangle$

vector space with an inner prod

Source: http://en.wikipedia.org/wiki/Space_(mathematics)

A hierarchy of mathematical spaces: The inner product induces a norm. The norm induces a metric. The metric induces a topology.

Topological space: Set of points along with a set of neighborhoods of each point, with certain axioms required to be satisfied by the pts & their nbrhoods

Metric space: Set of points with a notion of "distance" between elements $d(x,y)$ → must be ⓐ non-negative ⓑ $d(x,y)=0$ iff $x=y$ ⓒ symmetric ⓓ satisfy triangle inequality

Assuming you have understood vector space

Normed vector space: A vector space on which a norm is defined. (see page number 4 for definition of norm)

Definitions: In topological space, $\{x_i\}$ could converge to a limit $\lim_{i\to\infty} x_i$

- $\lim_{i\to\infty} \frac{1}{i} = 0$

//en.wikipedia.org/wiki/Limit_point

$cl(S)$ when S is a topological space

Should consist of S

union with

Should consist of $\lim_{i\to\infty} x_i$ for every convergent sequence $\{x_i\} \subseteq S$

For general topological space

S is closed if $cl(S) = S$

S is open if S^c is closed

$\text{interior}(S) = \bigcup_{\substack{S'=\text{open} \\ S'\subseteq S}} S'$

$bnd(S) = cl(S) - int(S)$

$\stackrel{?}{=} cl(S) \cap cl(S^c)$

with norm $\|\cdot\|$

$\forall x \in S, \exists \epsilon > 0$ s.t. $\{y \mid \|y-x\| \le \epsilon\} \subseteq S$ open set in Normed v.s

$bnd(S) = \partial(S)$

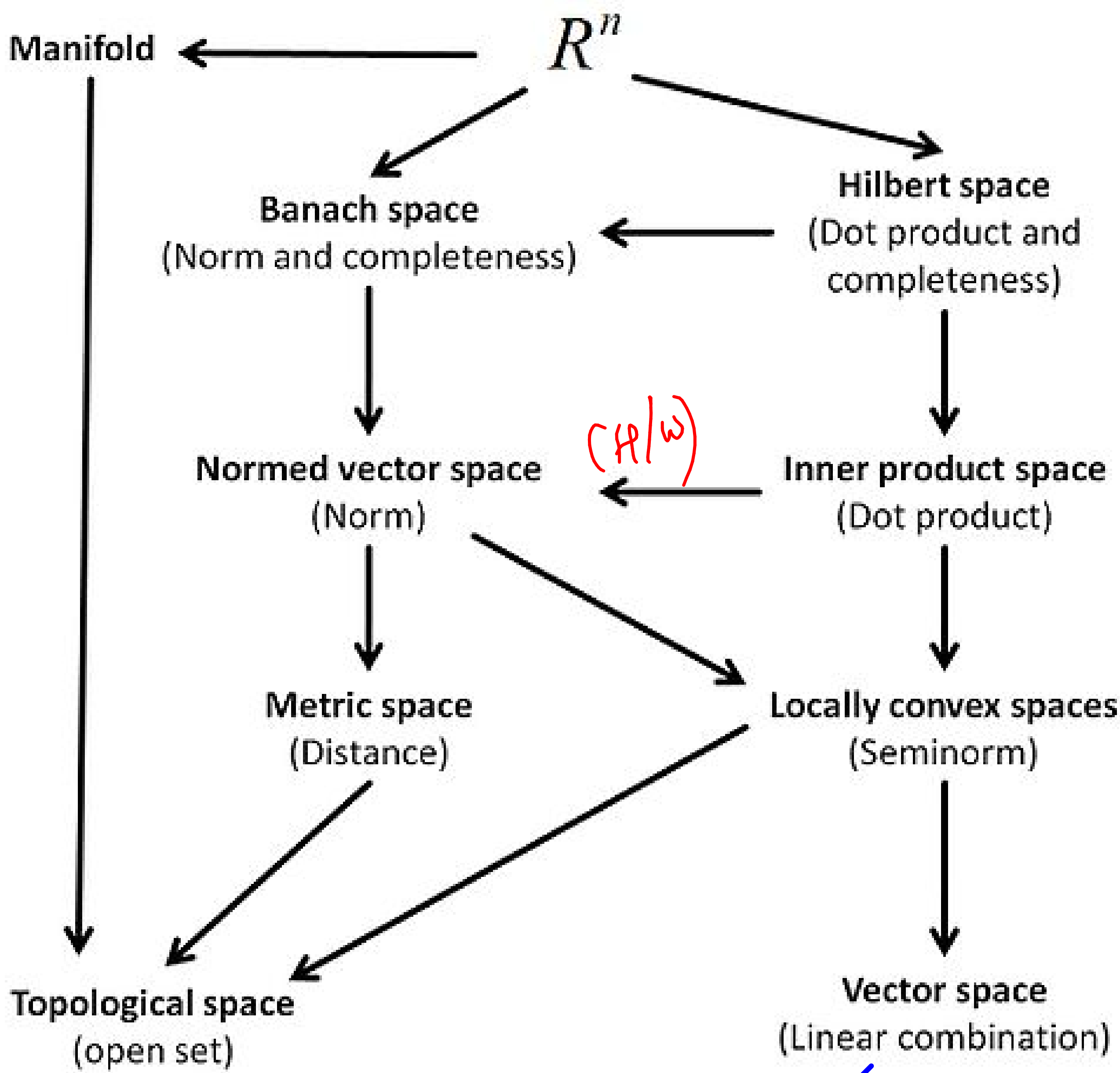
x belongs to the normed space

$cl(S) = \{x \mid \forall \epsilon > 0 \; S \cap \{y \mid \|x-y\| < \epsilon\} \ne \phi\}$

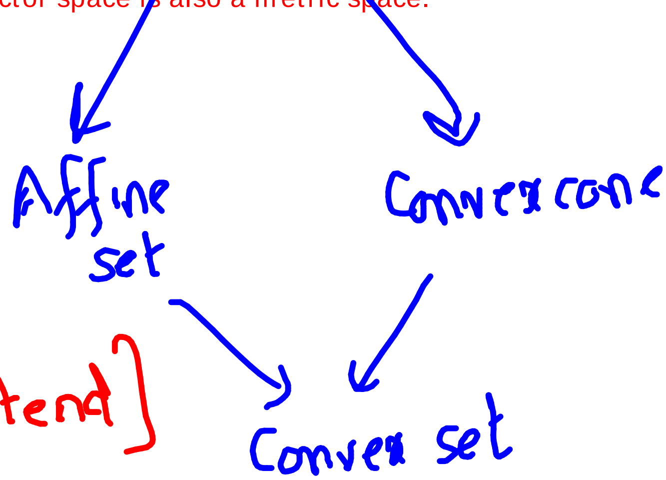
$int(S) = \{x \mid x \in S \text{ s.t } \exists \epsilon > 0 \text{ s.t } \{y \mid \|x-y\| < \epsilon\} \subseteq S\}$

$relbnd(S) = cl(S) - relint(S)$

$relint(S) = \{x \mid x \in S \text{ s.t } \exists \epsilon > 0 \text{ s.t } \{y \mid \|x-y\| < \epsilon\} \cap aff(S) \subseteq S\}$



Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



[H/W: Confirm or contend]

[H/w: Prove that "normed" space is a "metric" space]

Read how an inner product space is a normed space.

Inner product space: It is a vector space over a $\underbrace{\text{field of}}$ scalars along with an inner product

eg: $\mathbb{R}$

an algebraic structure.
with addition, subtraction, multiplication & division

associative & commutative

must be commutative associative distributive

multiplicative inverse must exist

Recall: $\|\cdot\|$ is a norm if

1. $\|x\| \geq 0$; $\|x\| = 0$ iff $x = 0$
2. $\|tx\| = |t| \|x\|$ for $t \in$ scalar
3. $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

a. (Conjugate) symmetry:
$\langle x, y \rangle = \langle y, x \rangle$

b. Linearity in the first argument
$\langle ax, y \rangle = a \langle x, y \rangle$
$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

c. Positive definiteness:
$\langle x, x \rangle \geq 0$ with equality iff $x = 0$

[H/W: Prove that "inner product space" is a "normed" vector space]

Inner product space: It is a vector space over a field of scalars along with an inner product

↓

Assume $\mathbb{R}$ or complex

① $\langle x, x\rangle = \overline{\langle x, x\rangle} \Rightarrow \langle x, x\rangle$ must be real

$\therefore$ We can define $\|x\| = \sqrt{\langle x, x\rangle}$

We need to prove that $\|x\|$ is a valid norm

ⓐ • By defn of inner product, since

$\langle x, x\rangle \geq 0$ with equality iff $x = 0$,

$\|x\| \geq 0$ iff $x = 0$

ⓑ • $\|tx\| = \sqrt{\langle tx, tx\rangle} = \sqrt{t \bar{t}\langle x, x\rangle}$

$= \sqrt{t \bar{t}}\, \|x\| = |t|\, \|x\|$ (For real & complex t, $|t| = \sqrt{t \bar{t}}$)

(c) $\|x+y\| = \sqrt{\langle x+y, x+y\rangle}$

$$= \sqrt{\langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle}$$

$$\leq \sqrt{\langle x,x\rangle + \langle y,y\rangle + \sqrt{\langle x,x\rangle\langle y,y\rangle} \times 2}$$

By Cauchy Schwarz inequality (proved on next page)

$$= \sqrt{(\|x\| + \|y\|)^2}$$

$$= \|x\| + \|y\|$$

– Hence proved that $\sqrt{\langle x,x\rangle}$ is a norm

normed v.s

ip v.s

$\Rightarrow$ Every inner product space is a normed space.

Converse does not hold: $\exists$ normed spaces that are not inner product spaces.

<u>Eg:</u> $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$

Eg: $p=1$ $\sum_{i=1}^{n} |x_i|$

$p=\infty$ $\max_{i=1..n} |x_i|$

Show that there does not exist $(x, y \in \mathbb{R}^n)$

$\langle x, y\rangle$ inner product s.t

$$\langle x, x\rangle = \left(\sum_{i=1}^{n} |x_i|^p\right)^{2/p}$$

V is a v.s

V is a normed vs

- scalar mult
- addition
- 0
- $\|\cdot\|$

Scalar mult

- addition
- 0

In general (see http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality)

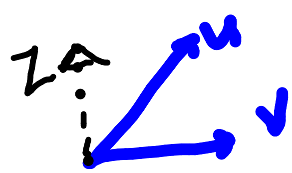
$$|\langle u,v\rangle| \le \|u\|\,\|v\| \quad \text{for} \quad \|u\|^2 = \langle u,u\rangle$$

Proof: If $v=0$, both sides are 0 & hence equality holds.

Assume $v \neq 0$ & let $z = u - \dfrac{\langle u,v\rangle}{\langle v,v\rangle} v$ $\Big\}$ $z=0$ iff u & v are lin. dependent

& ∴ $$\langle z,v\rangle = \left\langle u - \frac{\langle u,v\rangle}{\langle v,v\rangle} v, v\right\rangle = \langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle}\langle v,v\rangle = 0$$

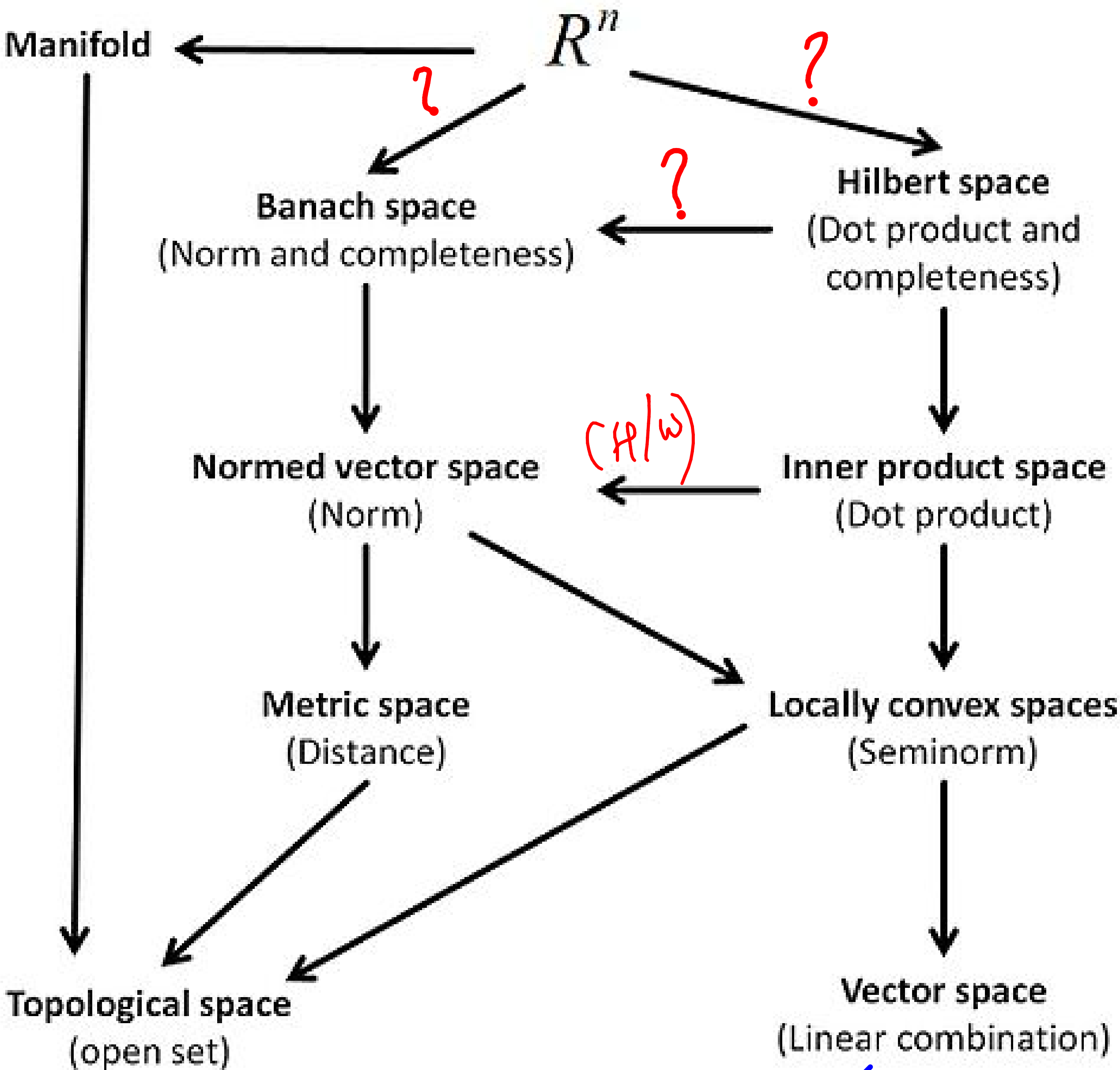
(By linearity of the inner product in the first argument)



& ∴ $$\langle u,u\rangle = \|u\|^2 = \underbrace{\langle z,z\rangle + \left|\frac{\langle u,v\rangle}{\langle v,v\rangle}\right|^2 \langle v,v\rangle}_{\text{Substituting for } u = z + \frac{\langle u,v\rangle}{\langle v,v\rangle} v} + \underbrace{\left|\frac{\langle u,v\rangle}{\langle v,v\rangle}\right| \langle z,v\rangle}_{=0 \text{ from above}}$$

$$= \|z\|^2 + \left(\frac{\langle u,v\rangle^2}{\|v\|^2}\right) \ge \frac{\langle u,v\rangle^2}{\|v\|^2}$$ {equality iff $z=0$}

$\Rightarrow \boxed{\|u\|\,\|v\| \ge |\langle u,v\rangle|}$ → Cauchy Shwarz ineq.
{equality iff u & v are linearly dependent}



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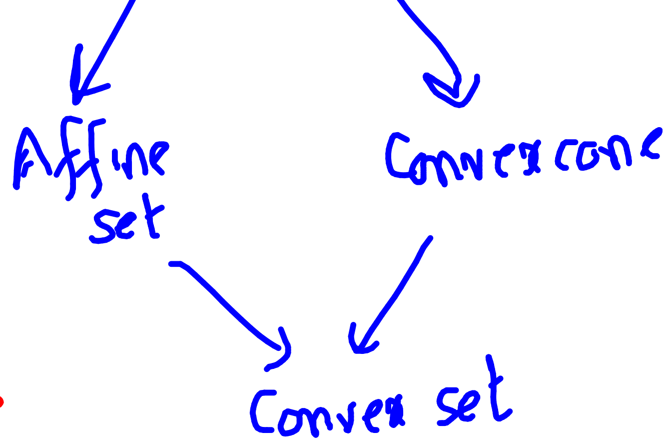


Recap:

Topological space S is closed if S contains limit points of all convergent sequences in S

Cauchy sequence: (in any metric space)

A sequence is cauchy if its all its terms "eventually" become arbitrarily close to one another.

↳ ie given $\epsilon > 0$, $\exists$ N s.t if for any $m, n > N$ then $d(a_m, a_n) < \epsilon$

Q: Which of the following sequences are (a) Cauchy (b) Convergent?

(i) $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$ in $\mathbb{R}$ (a) Cauchy & (b) Convergent

(ii) $\left(1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \dots\right) = \left(\dots \sum_{i=1}^{k} \left(\frac{1}{i}\right) \dots\right)$ in $\mathbb{R}$

(b) $a_k \geq 1 + (\log_2 k)\ r(k) \xrightarrow[k \to \infty]{} \infty$ $\therefore$ Not convergent

(a) Not cauchy since $\mathbb{R}$ & not convergent?

(iii) $\left(1, \frac{1+\frac{2}{1}}{2}, \dots x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2} \dots\right)$ in $\mathbb{Q}$

(b) It is convergent in $\mathbb{R}$ (to $\sqrt{2}$) but NOT in $\mathbb{Q}$

(a) It is cauchy because it is convergent

Read: Babylonian method & Maclaurin series → for approx $\sin(x)$, $\cos(x)$

(iv) $\left(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n} \dots\right)$ in $(0, 2)$

(a) Cauchy by (i) (b) Not convergent in $(0, 2)$

Idea: If x is an overestimate of $\sqrt{m}$ where m is a non-negative real no, then $\frac{m}{x}$ is an underestimate of the square root:

ie if $x \geq \sqrt{m}$ then $\frac{m}{x} \leq \sqrt{m}$

$1.5 \geq \sqrt{2}$ $\quad$ $2/1.5 \leq \sqrt{2}$

$\Rightarrow \frac{1}{2}\left(x + \frac{m}{x}\right)$ will tend to approximate $\sqrt{m}$

Idea: $m = (x+e)^2 \Rightarrow e = \frac{m - x^2}{2x+e} \approx \frac{m-x^2}{2x}$ (if $e \ll x$)

$$\Rightarrow \boxed{x_{n+1} = x_n + e = \frac{x_n + \frac{m}{x_n}}{2}}$$

Claim: Any convergent sequence in a Metric space must be cauchy

Proof:

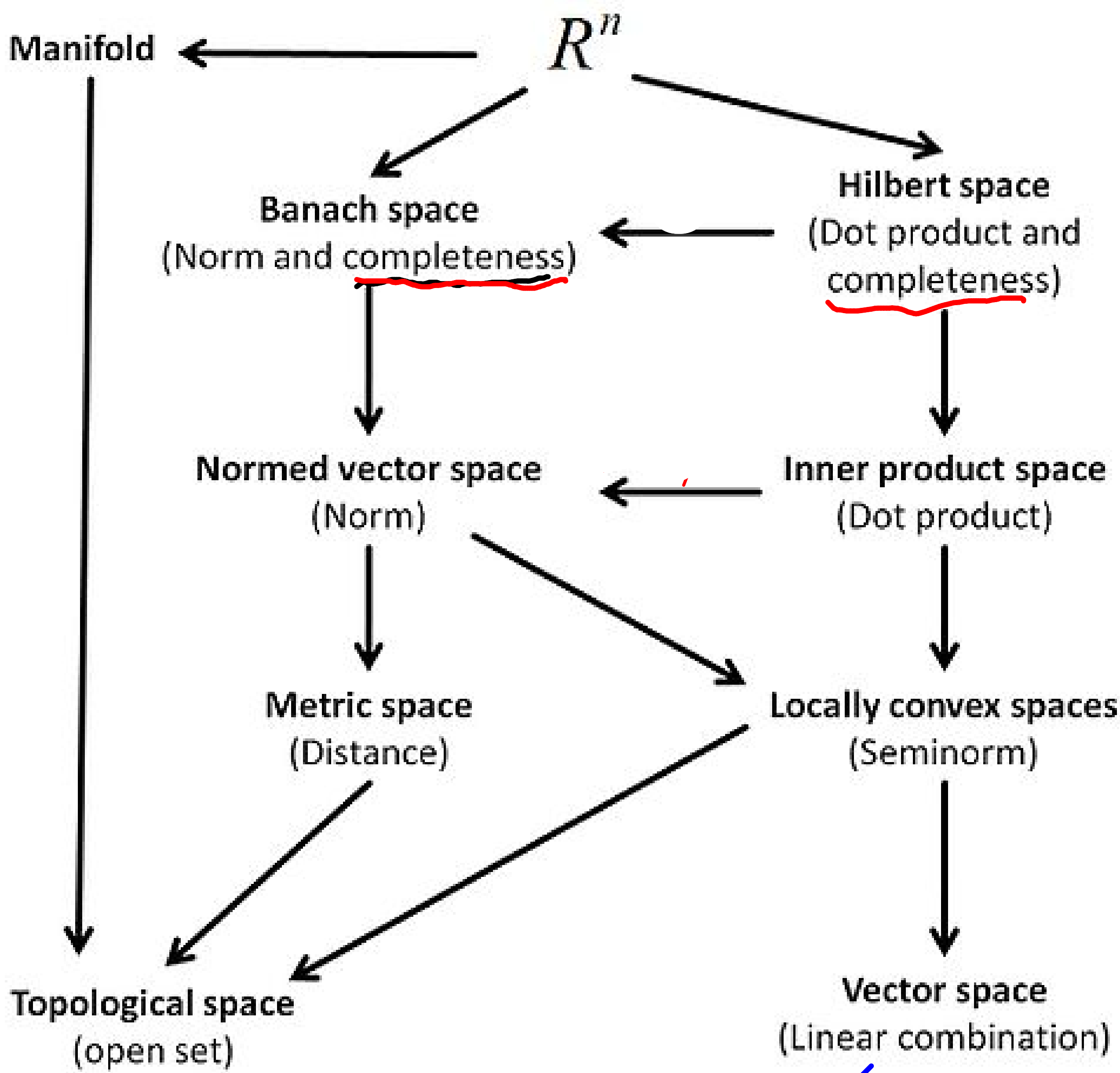
Let $(s_n) \to s$. Given $\epsilon > 0$ choose N s.t if $n > N$, we have $d(s_m, s_n) < \epsilon$

Then if $m, n > N$, $d(s_m, s_n) \le d(s_m, s) + d(s, s_n)$

$< 2\epsilon$

BUT GIVEN A METRIC SPACE S, EVERY CAUCHY SEQUENCE NEED NOT CONVERGE TO A LIMIT POINT IN S!

(We saw several examples: $\left(x_{n+1} = \dfrac{x_n + \frac{2}{x_n}}{2}\right)$



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Complete metric space

A metric space S in which every cauchy sequence in S is convergent in S

Affine set

Convex cone

Convex set

Specialities of $\mathbb{R}^n$

① Every cauchy sequence is convergent

② A bounded sequence has atleast one limit point : Bolzano Weierstrass Theorem

Eg: $(1, 0, 1, 0, 1 \ldots.)$

$x \in \mathbb{R}^n$ is said to be a limit point of $\{x_k\}$ if $\exists$ a subsequence of $\{x_k\}$ that converges to x.