Claim: Any convergent sequence in a Metric space must be cauchy

Claim: Every cauchy sequence is bounded

Claim: A bounded sequence in $\mathbb{R}^n$ has atleast one limit point: Bolzano Weierstrass Theorem

Eg: $(1, 0, 1, 0, 1 \ldots)$

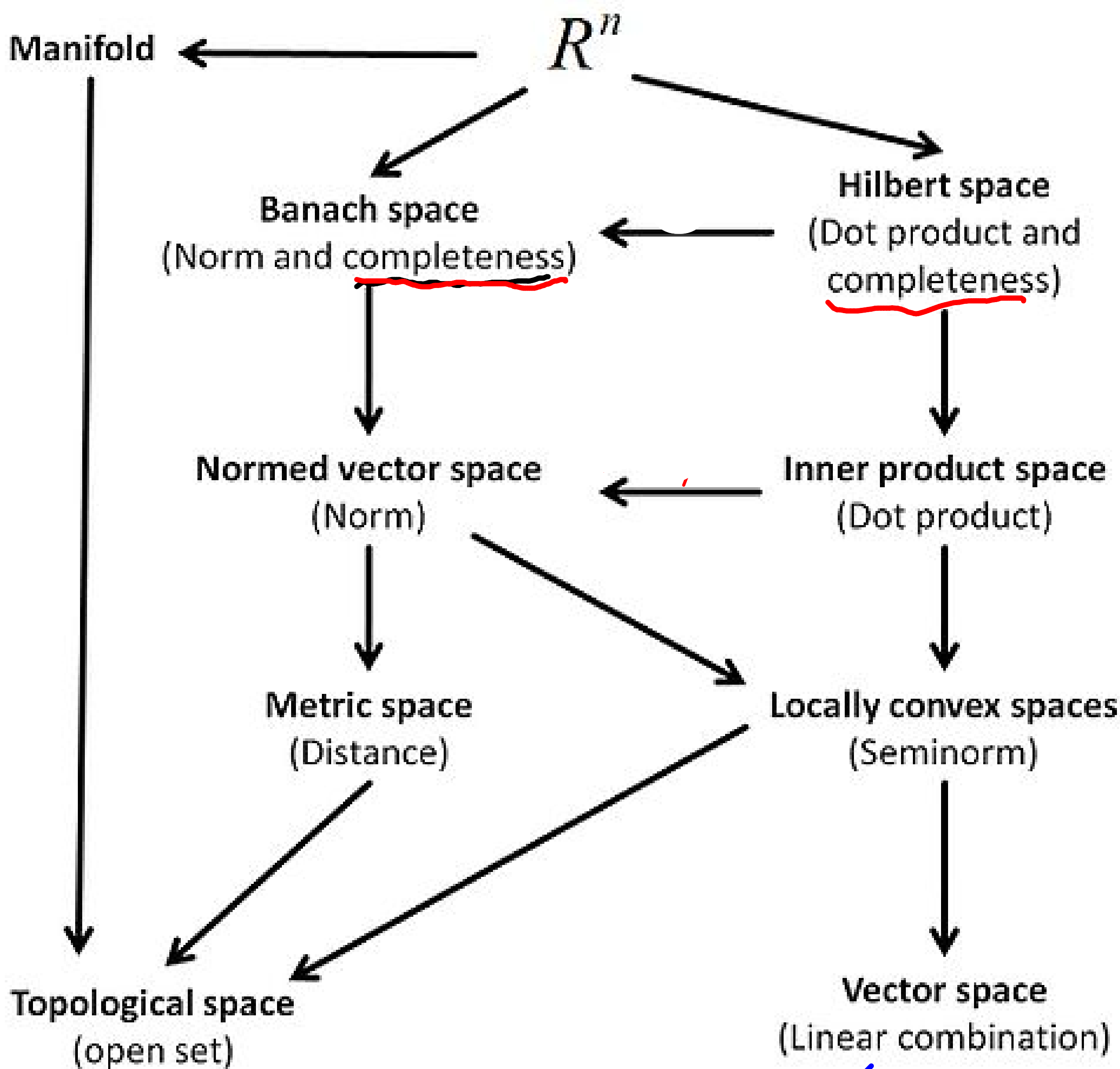
$x \in \mathbb{R}^n$ is said to be a limit point of $\{x_k\}$ if $\exists$ a subsequence of $\{x_k\}$ that converges to $x$.

Claim: GIVEN A METRIC SPACE S, EVERY CAUCHY SEQUENCE NEED NOT CONVERGE TO A LIMIT POINT IN S!

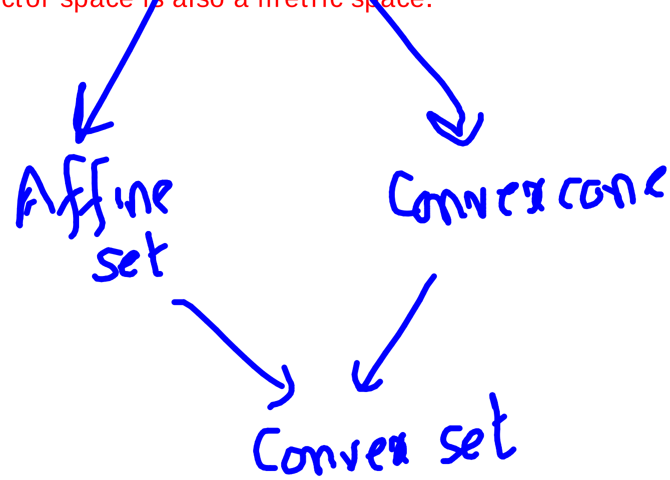
(we saw several examples: $\left(x_{n+1} = \dfrac{x_n + \frac{2}{x_n}}{2}\right)$

Claim: In $\mathbb{R}^n$, every Cauchy sequence converges to a limit point in $\mathbb{R}^n$

Such spaces are called complete spaces




Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.




Complete metric space

A metric space S in which every cauchy sequence in S is convergent in S

Show that the following are vector spaces (assuming scalars come from a set S), and then answer questions that follow for each of them: Set of all matrices on S, set of all polynomials on S, set of all sequences of elements of S. (HINT: You can refer to this book for answers to most questions in this homework.) How would you understand the concepts of independence, span, basis, dimension and null space (chapter 2 of this book), eigenvalues and eigenvectors (chapter 5), inner product and orthogonality (chapter 6)? EXTRA: Now how about set of all random variables and set of all functions. **Deadline:** January 23 2015.

Examples: Let $S$ be a field. Good examples can be found at http://en.wikipedia.org/wiki/Field_(mathematics)#Examples

(a) $S^\infty$: Space of infinite sequences of elements from $S$ : $(x_1, x_2, x_3 \ldots\ldots)$

↳ Only finitely many non-zero elements

$\Rightarrow$ Basis $= \{\underbrace{(1, 0 \ldots)}_{e_1}, \underbrace{(0, 1, 0, 0 \ldots)}_{e_2}, \underbrace{(0 \ldots 1, 0 \ldots)}_{e_i} \ldots\}$

$x \in \text{Span}(V)$ if $x$ can be obtained by linear combination of finite # of elements from $V$

Dimensionality = countably infinite

↳ No restriction on non-zero elements

$\Rightarrow$ Basis exists (enumerating basis is open)

Dimensionality = uncountably infinite

(Banach space) $\ell^p$ ↳ With bounded p-norm: $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty$ $p > 0$

Dimensionality = countably infinite

Eg: For $p=2$ you have $\ell^2$: Square summable sequences

$$\ell^p = \left\{ (x_1, x_2 \ldots) \,\middle|\, \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}$$

$$\boxed{\text{Note: } \quad \ell^p \subseteq \ell^{p'} \quad \text{for } p' > p}$$

Does this hold for $\mathbb{R}^n$: Yes with $=$

$$\sum_{i=1}^{\infty} \left|\frac{1}{i}\right|$$

Eg: $\left(1, \frac{1}{2}, \ldots \frac{1}{n}, \frac{1}{n+1} \ldots\right) \notin \ell^1$

But: $\left(1, \frac{1}{2}, \ldots \frac{1}{n}, \frac{1}{n+1} \ldots\right) \in \ell^p$ for $p > 1$ $\quad \left( \sum_{i=1}^{\infty} \left|\frac{1}{i}\right|^p \right)^{1/p}$

Q: Is $\ell^p$ Hilbert space?

Ans: Only when $p=2$

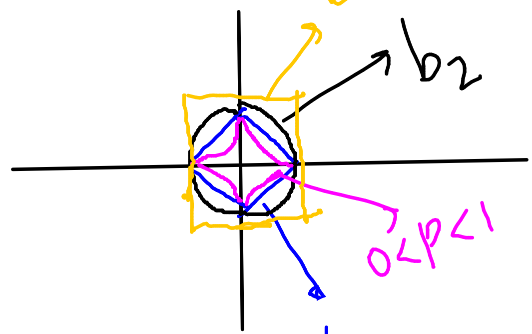
Aside

Q: For $\mathbb{R}^n$

$$\{\|x-x_0\|_p \le r\} \overset{?}{\subseteq} \{\|x-x_0\|_{p'} \le r\}$$

$p' > p$

Ans: Yes



Show that there does not exist $\langle x,y\rangle$ inner product s.t $(x,y\in\mathbb{R}^n)$

$$\langle x,x\rangle = \left(\sum_{i=1}^{n} |x_i|^p\right)^{2/p}$$

V is a v.s

V is a normed vs
scalar mult
addition
0
$\|\cdot\|$

Scalar mult
addition
0

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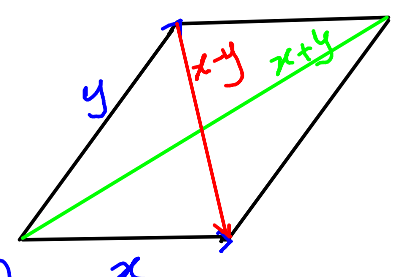
Solution:

If $\|x\|$ were defined using an inner product $\sqrt{\langle x,x\rangle}$ then the following should hold (also called the parallelogram law)

$$\|x\|^2+\|y\|^2 = \langle x,x\rangle+\langle y,y\rangle$$

$$= \frac{1}{2}\underbrace{\big(\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle\big)}_{\|x+y\|^2} + \frac{1}{2}\underbrace{\big(\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle\big)}_{\|x-y\|^2}$$

$$= \frac{1}{2}\left(\|x+y\|^2+\|x-y\|^2\right)$$




Now: Let $x=[a,a,0\ldots0]$ $y=[a,-a,0\ldots0]$

Then: $\|x+y\|_p = \left\| \begin{bmatrix} 2a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_p = \left(|2a|^p\right)^{1/p} = 2|a|$

$\|x-y\|_p = \left\| \begin{bmatrix} 0 \\ 2a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_p = \left(|2a|^p\right)^{1/p} = 2|a|$

$\|x\|_p = \left\| \begin{bmatrix} a \\ a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_p = \left(|a|^p + |a|^p\right)^{1/p} = 2^{1/p} * |a|$

$\|y\|_p = \left\| \begin{bmatrix} a \\ -a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_p = \left(|a|^p + |-a|^p\right)^{1/p} = 2^{1/p} * |a|$

For the parallelogram law to be satisfied

$$\underbrace{2 * 2^{2/p} |a|^2}_{\|x\|_p^2 + \|y\|_p^2} = \underbrace{\frac{1}{2} * 2 * 2^2 |a|^2}_{\frac{1}{2}\left(\|x+y\|_p^2 + \|x-y\|_p^2\right)}$$

i.e $\quad 2 * 2^{2/p} = 4$

i.e $\quad \boxed{2^{2/p} = 2 \quad \text{which holds iff} \quad p = 2}$

Thus: $\|x\|_p$ corresponds to an inner product $\sqrt{\langle x, x \rangle}$

iff $p = 2$

iff since $\exists \sqrt{\langle x, x \rangle} = \|x\|_2$

where $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$

## (b) Polynomial vector spaces

$P^n(x)$ = set of polynomials with coefficients in $S$ with degree $\leq n$ in a single variable $x = \{\sum_{i=0}^{n} s_i x^i \mid s_i \in S\}$

↳ Can be viewed as vector space $S^{n+1}$ and dimension $= n+1$

$P(x)$ = Set of polynomials with coefficients in $S$ with no degree restriction but in a single variable $x$

↳ Can be viewed as vector space of infinite sequences... All discussions (norm, inner product etc follow)

$P(x_1 .. x_k)$ = Set of polynomials in $k$ variables

↳ Result similar to $P(x) / P^n(x)$ ?

↳ How about finite/infinite sequences of matrices

$$P(x_1 .. x_k) = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \end{bmatrix}$$

③ Function spaces: $f: X \to V$ ($V$ is vector space over $S$)

↳ If $X$ is finite & $V$ is finite dimensional, then $f$ has dimension $|X|\dim(V)$

Basis $= \{f_{ij}\}$

$f_{ij}(x_i) = v_j$
$f_{ij}(x_k) = 0$

$\{v_1 \ldots v_t\} = \text{Basis}(V)$

↳ If $X$ is finite & $V$ is countably infinite dimensional then $\dim f$ is countably infinite (similarly construct basis)

↳ Else $\dim f$ is uncountably infinite (Prove H/W)

↳ Normed spaces:

$$L^p = \left\{ f \;\middle|\; f: X \to S \,.\; \|f\|_p = \left( \int_X |f(x)|^p \, dx \right)^{1/p} < \infty \right\}$$

Should be structure preserving between domain & range ie measurable

$S = \mathbb{C}$ or $\mathbb{R}$

$L^p$ is Banach for $p \geq 1$

$L^p$ is Hilbert only for $p = 2$

$$\langle f, g \rangle = \int f(x) \overline{g(x)} \, dx$$

# ④ Vector space of mxn matrices

- Show that the following are vector spaces (assuming scalars come from a set S), and then answer questions that follow for each of them: Set of all matrices on S, set of all polynomials on S, set of all sequences of elements of S. (HINT: You can refer to this book for answers to most questions in this homework.) How would you understand the concepts of independence, span, basis, dimension and null space (chapter 2 of this book), eigenvalues and eigenvectors (chapter 5), inner product and orthogonality (chapter 6)? EXTRA: Now how about set of all random variables and set of all functions.

Let us consider space of matrices:

$$\left\{ \begin{bmatrix} s_{11} & \cdots & s_{1m} \\ & & \\ s_{n1} & \cdots & s_{nm} \end{bmatrix} \middle| \; s_{11} \cdots s_{nm} \in S \right\}$$ over scalars S

So far we considered $S = \mathbb{R}$

Obvious that this is a vector space (since multiplication etc are defined on S)

For simplicity, let $S = \mathbb{R}$ & let us consider a norms for matrices, induced by norms for vectors

Let $N(x)$ be a vector norm satisfying the vector norm axioms:

(Define $\|A\| = f(A, N(x))$ for any/all $x \in \mathbb{R}^n$ )

Then we will define a matrix norm

$$M_N(A) = \sup_{x \neq 0} \frac{N(Ax)}{N(x)}$$

Can you prove that this is indeed a valid norm?

$\sup_{S} f(s) = \hat{f}$ if $\hat{f}$ is minimum upper bnd

as the matrix norm induced by vector norm $N(x)$

What, for example, will be $M_N(I)$ irrespective of $N(x)$? → Ans: 1

$$\left( \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right)$$

examples

a) If $N(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$

$$\|A\|_1 = ?$$

Ans: $\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}| |x_j|$

abs value of sum $\leq$ sum of abs values

Changing order of summation:

$$\|Ax\|_1 \leq \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$$

→ Let $C = \max_j \sum_{i=1}^{n} |a_{ij}|$

Then $\|Ax\|_1 \le C\|x\|_1$

$$\Rightarrow \quad \|A\|_1 = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \le C$$

But consider an $x = [0 \, . \, 0 \, . \, 1 \, . \, . \, 0 \, 0]$

$k^{th}$ position, where $k$ is column index $j$ for which $C = \sum_{i=1}^{n} |a_{ik}|$

Then $\|x\|_1 = 1$ & $\|Ax\|_1 = C$ (Show this)

$$\Rightarrow \quad \|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

i.e. if $N(x) = \|x\|_1$ then $M_N(A) = \max_j \sum_i |a_{ij}|$

(b) Similarly,

if $N(x) = \|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$

$$\|A\|_2 = \left[\text{dominant eigenvalue of } A^TA\right]^{1/2}$$

(c) If $N(x) = \|x\|_\infty = \max_i |x_i|$ $\left\{\lim_{p\to\infty} \left(\sum |x_i|^p\right)^{1/p}\right.$

$$\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$$

Other matrix norms:

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$$

Frobenius norm $= \sqrt{\mathrm{Trace}(A^T A)}$

→ special case

① If $A$ is symmetric: $\|A\|_F^2 = \lambda_1(A)^2 + \lambda_2(A)^2 + \dots + \lambda_n(A)^2$

② If $A$ is not symmetric $\|A\|_F^2 = \sigma_1(A)^2 + \dots + \sigma_k(A)^2$

rank$(A) = k$.

(By virtue of trace)

Q: What abt inner products:

Note: Not all normed spaces are inner prod spaces.

Eg: $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ → for $p=2$ $\langle x,y\rangle = \sum_i x_i y_i$

→ For $p = 1$ or $\infty$, No corresp. inner products

Read more on

http://www.math.ucsd.edu/~njw/Teaching/Math271C/Lecture_04.pdf

Eg of Frobenius inner product:

$$\langle A,B\rangle = \sum_i \sum_j a_{ij} b_{ij}$$

$$\langle A,B\rangle_W = \sum_i \sum_j a_{ij} b_{ij} w_{ij} \quad \text{for } w_{ij} > 0$$

weighted inner product

Singular values & Eigenvalues of A

$\sigma$ $\lambda$

$Au = \sigma v$

$A^* v = \sigma u$

$\Downarrow$

$A^* A u = \sigma^2 u$

ie $\sigma^2$ is an eigenvalue of $A^* A$

$Au = \lambda u$

$$\boxed{\begin{array}{l} \text{If } A^T = A \text{ or } A^* = A \\ A^* A u = \lambda^2 u \\ \qquad\quad = \sigma^2 u \end{array}}$$

Basis for vector space of matrices $(m \times n)$

$$\left\{ \underbrace{\begin{bmatrix} 1 & 0 & \cdot & 0 \\ \cdot & & & \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}}_{E_{11}} \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \cdots \\ & & & \\ 0 & & \cdots & 0 \end{bmatrix}}_{E_{12}} \cdots \underbrace{\begin{bmatrix} 0 & \cdots & & 0 \\ \cdot & & & \\ \vdots & & & \\ 0 & \cdots & & 1 \end{bmatrix}}_{E_{mn}} \right\}$$

$m \times n$ linearly independent elements that span the space of all matrices of size $m \times n$

$$B = \sum_{i,j} a_{ij} E_{ij} \equiv \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{mn} \end{bmatrix}$$

This vector $\in \mathbb{R}^{mn} / S^{mn}$ is a canonical representation of $B$

## Euclidean balls and ellipsoids

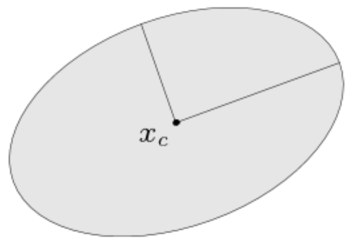
**(Euclidean) ball** with center $x_c$ and radius $r$:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

**ellipsoid:** set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ ($i.e.$, $P$ symmetric positive definite)




other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with $A$ square and nonsingular



## Norm balls and norm cones

**norm:** a function $\|\cdot\|$ that satisfies

- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

**norm ball** with center $x_c$ and radius $r$: $\{x \mid \|x - x_c\| \le r\}$

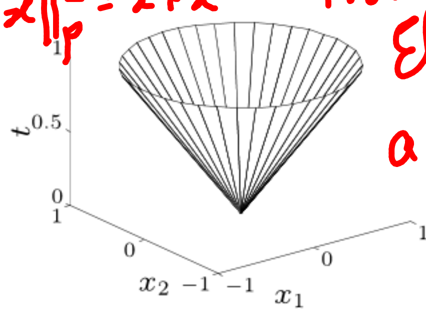
Eucledian ball → $\| \ \|_2$

Ellipsoid ⇒ $\|x\|_P^2 = x^T P x$

Prove that Ellipsoid is a norm ball

**norm cone:** $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone




norm balls and cones are convex