Euclidean balls and ellipsoids

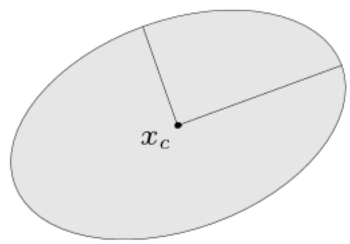
(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \le r\}$

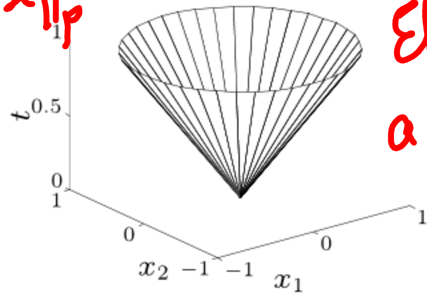
Prove that Ellipsoid is a norm ball

Eucledian ball $\Rightarrow \| \ \|_2$

Ellipsoid $\Rightarrow \|x\|_P^2 = x^T P x$

norm cone: $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

$\therefore$ ① $x^T P x \geq 0$ since P is positive definite

$\&$ $x^T P x = 0$ iff $x = 0$ (By definition)

② $\|\alpha x\|_P = \sqrt{(\alpha x)^T P (\alpha x)} = \sqrt{\alpha^2 x^T P x}$

$= |\alpha| \, \|x\|_P$

③ $\|x+y\|_P^2 = (x+y)^T P (x+y) = (x+y)^T R R^T (x+y)$

$= x^T R \underbrace{R^T x}_{u} + y^T R \underbrace{R^T y}_{v} + x^T R R^T y$

$+ y^T R R^T x$

$= u^T u + v^T v + u^T v + v^T u$

$= \|u\|_2^2 + \|v\|_2^2 + 2u^T v$

$(\|x\|_P + \|y\|_P)^2 = \underbrace{\|x\|_P}^2 + \underbrace{\|y\|_P} + 2\|x\|_P \|y\|_P$

$= \|u\|_2^2 + \|v\|_2^2 + 2\sqrt{\|u\|_2^2 \|v\|_2^2}$

$= \|u\|_2^2 + \|v\|_2 + 2\|u\|_2 \|v\|_2$

Next, Use the Cauchy Schwarz inequality:

$2u^T v \leq \|u\|_2 \|v\|_2 \Rightarrow \|x+y\|_P \leq \|x\|_P + \|y\|_P$

Prove that under specific assumptions on P, $\sqrt{x^T P x}$ is a valid norm. Assume $x \in \mathbb{R}^n$ & $P \in \mathbb{R}^{n \times n}$

Proof: Suppose P is symmetric positive definite:

i.e $P^T = P$ & $\forall x \neq 0 \; x^T P x > 0$

The condition $\forall \, \mathbf{x} \neq \mathbf{0}, \sum_{i=1} \sum_{j=1} a_{ij} \mathbf{x}_i \mathbf{x}_j > 0$ involves a quadratic expression. The expression is guaranteed to be greater than $0 \; \forall \, \mathbf{x} \neq \mathbf{0}$ *iff* it can be expressed as $\sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{i-1} \beta_{ij} x_{ij} + x_{ii} \right)^2$, where $\lambda_i \geq 0$. This is possible *iff* A can be expressed as LDL^T, where, L is a lower traingular matrix with 1 in each diagonal entry and D is a diagonal matrix of all positive diagonal entries. Or equivalently, it should be possible to factorize A as RR^T, where $R = LD^{1/2}$ is a lower traingular matrix. Note that any symmetric matrix A can be expressed as LDL^T, where L is a lower traingular matrix with 1 in each diagonal entry and D is a diagonal matrix; positive definiteness has only an additional requirement that the diagonal entries of D be positive. This gives another equivalent condition for positive definiteness: *Matrix A is p.d. if and only if, A can be uniquely factored as $A = RR^T$, where R is a lower traingular matrix with positive diagonal entries.* This factorization of a p.d. matrix is reffered to as *Cholesky factorization.*

Source: pg 207 of http://www.cse.iitb.ac.in/~CS709/notes/LinearAlgebra.pdf

$$\Rightarrow x^T P x = x^T \underbrace{R R^T}_{\text{Assume } P = RR^T} x = \underbrace{(R^T x)^T}_{\text{Let } R^T x = y} (R^T x) = y^T y = \|y\|_2^2$$

Prove that under specific assumptions on P, $\sqrt{x^T P x}$ is a valid norm. Assume $x \in \mathbb{R}^n$ & $P \in \mathbb{R}^{n \times n}$

Proof: Suppose P is symmetric positive definite:

i.e $P^T = P$ & $\forall x \neq 0 \;\; x^T P x > 0$

① $\|x\|_P^2 = x^T P x \geq 0$ with equality iff $x = 0$ (obvious)

② $\|\alpha x\|_P = \sqrt{\alpha^2 x^T P x} = |\alpha| \sqrt{x^T P x} = |\alpha| \, \|x\|_P$

③ $\|x+y\|_P^2 = (x+y)^T P (x+y) = x^T P x + y^T P y + 2 x^T P y$

(since $P = P^T \Rightarrow x^T P y = y^T P x = (x^T P y)^T$)

$$= \|x\|_P^2 + \|y\|_P^2 + 2 x^T P y$$

(need to prove) $\leq \|x\|_P^2 + \|y\|_P^2 + 2 \|x\|_P \|y\|_P$

Using positive definiteness of P, can you prove that

$$x^T P y \leq \sqrt{x^T P x} \sqrt{y^T P y} \; ?$$

$l_p(S)$ = set of sequences with scalar field S
that have finite p-norm
$$\|x\|_p = \left(\sum_{i=1}^{\infty} x_i^p\right)^{1/p}$$

$L_p(S)$ = set of measurable functions
$f: X \to S$ with finite p-norm
$$\|f\|_p = \left(\int_{x \in X} f^p(x)\,dx\right)^{1/p}$$

A linear map/linear operator T between two vector spaces X & Y is $T: X \to Y$ s.t

$$T(\lambda x + \mu x') = \lambda Tx + \mu Tx'$$

$\forall\ \lambda, \mu \in S$
$\forall\ x, x' \in X$

If T is 1-1 & onto then T is called invertible. T^{-1} is defined s.t

$T^{-1}: Y \to X$ s.t $T^{-1}y = x$ iff $Tx = y$

(a) if X & Y are normed spaces

$$\|T\| = \sup_{\substack{x \neq 0 \\ x \in X}} \frac{\|Tx\|}{\|x\|}$$ is called operator norm

T is called bounded if

$\exists N$ s.t $\|T\| \leq N \iff \exists M$ s.t $\|Tx\| \leq M\|x\|$ $\forall x \in X$

T is bounded iff it is continuous

Proof: Recall that T is called continuous if given any $\epsilon > 0$, $\exists$ a $\delta > 0$ st whenever

$\|x - x'\| \leq \delta$. $\|Tx - Tx'\| \leq \epsilon$ $x, x' \in X$

(a) Suppose $T: X \to Y$ is bounded. Then $\forall x, x' \in X$ we have

$$\|Tx - Tx'\| = \|T(x - x')\| \leq M\|x - x'\|$$

where M is s.t $\|Tx\| \le M\|x\| \ \forall x \in X$
Taking $\delta = \epsilon/M$, we get that T is cts

(b) Suppose $T: X \to Y$ is continuous at all $x \in X$ including $0 \in X$. T being linear, we must have $T(0) = 0$. Let $\epsilon = 1$, then $\exists$ $\delta > 0$ s.t $\|Tx\| \le 1$ whenever $\|x\| \le \delta$.
For any $x \in X$ s.t $x \neq 0$, let $\tilde{x} = \delta \dfrac{x}{\|x\|}$
Now $\|\tilde{x}\| \le \delta \Rightarrow \|T\tilde{x}\| \le 1$. From linearity of T: $\|Tx\| = \dfrac{\|x\|}{\delta}\|T\tilde{x}\| \le M\|x\|$
where $M = \dfrac{1}{\delta}$. Thus, T is bounded

(b) If $X = \{f: D \to V\}$ is a space of functions from domain D to vector space V & $T: X \to X$ then f is called an eigenfunction of T & λ corresponding eigenvalue if

$$Tf = \lambda f$$

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(b) If $X = \{f : D \to V\}$ is a space of functions from domain D to vector space V & $T: X \to X$ then f is called an eigenfunction of T & λ corresponding eigenvalue if

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(c) If $T: X \to Y$ & T is 1-1 & onto then X and Y are said to be

LINEARLY ISOMORPHIC

If X & Y are Hilbert spaces then they are isomorphic if $\exists$ an orthogonal (unitary) linear map $U: X \to Y$.

That is U should satisfy

$$\langle x, x' \rangle_X = \langle Ux, Ux' \rangle_Y \quad \forall \; x, x' \in X$$

(d) If $T: X \to Y$

kernel of $T = \ker(T) = \{ x \in X \mid Tx = 0 \}$

range of $T = \operatorname{ran}(T) = \{ y \in Y \mid \exists \; x \in X \text{ s.t } Tx = y \}$

Both kernel and range are vector spaces

If X & Y are normed & T is bounded then $\ker(T)$ is closed

(e) If X is normed v.s & Y is Banach then $T: X \to Y$ is Banach w.r.t the operator norm

(f) $T: X \to \mathbb{R}$ is called a linear functional

Then dual of X is set of all its linear functionals

Algebraic dual $= \{T \mid T: X \to \mathbb{R}\} = X^{\#}$

If X is finite dimensional, its dual $X^{\#}$ is linearly isomorphic to X

ie if $\{e_1 \ldots e_n\}$ is basis for X

then $\{g_i : X \to \mathbb{R}\}$ s.t

$$g_i\left(\sum_{j=1}^{n} x_j e_j\right) = x_i$$

form the basis for $X^{\#}$ so that

for any $g \in X^{\#}$ s.t $g\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} g(e_i) x_i$

$\{g_1, g_2 \ldots g_n\}$ is called the dual basis

9. Topological dual $= \{T \mid T: X \to \mathbb{R}\} = X^*$

In finite dimensional case: $X^* = X^\#$
& X^* is isomorphic to X

T is a continuous linear functional

You get specific duals for subsets of vector spaces (such as convex sets, cones and affine sets) by putting restrictions on T.

Eg: If $C \subseteq X$ s.t X is a vector space

(a) $C^\#$ = algebraic dual cone
$= \{T \in X^\# \mid T(x) \geq 0 \ \forall x \in C\}$

(b) Further if X is a topological vector space & $C \subseteq X$

then

$C^* =$ topological dual cone

$= \{ T \in X^* \mid T(x) \geq 0 \ \forall x \in C \}$

Claims:

① C^* is always a convex cone (irrespective of whether C is convex cone or neither)

if $T_1 \in C^*$ & $T_2 \in C^*$ & $\theta_1, \theta_2 \geq 0$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \in X^*$$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \geq 0$$

$\Rightarrow C^*$ is a convex cone

(Similarly $C^\#$ is also always a convex cone)

② If X is finite dimensional,

$C^{\#} = C^{*}$

since $X^{\#} = X^{*}$

③ If X is a Hilbert space,

C^{*} is closed .... More properties follow when X is a Hilbert space

(h) Riesz representation theorem:

If $T : X \to \mathbb{R}$ and X is Hilbert and T is bounded, then $\exists$ a unique vector $y \in X$ s.t

$$T(x) = \langle y, x \rangle \quad \forall\, x \in X$$

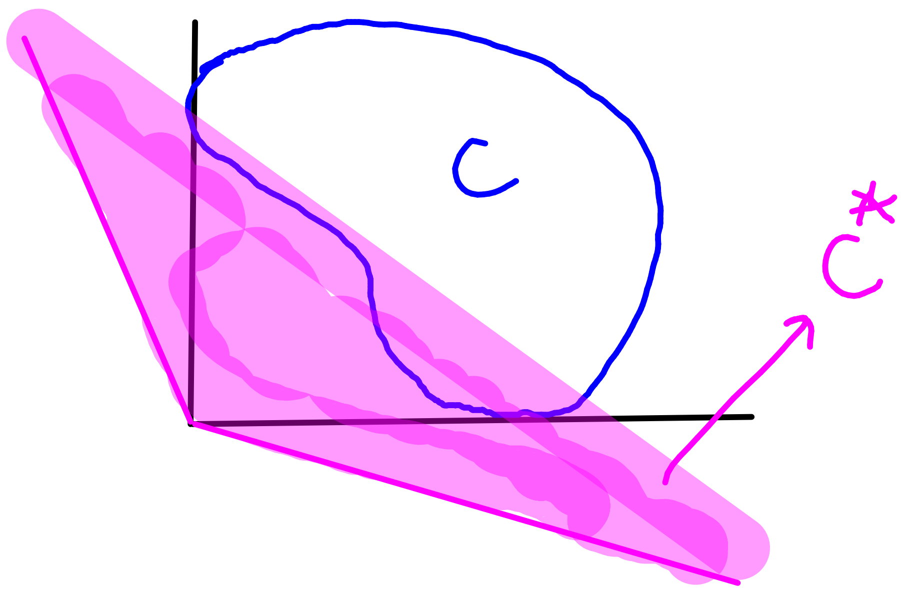
In fact $X^* = \{ T_y(x) = \langle y, x \rangle \mid x \in X \}$ is the dual of X^*

Defines a linear functional in terms of an inner product

Further, X & X^* are isomorphic.

(i) Thus, if X is a Hilbert space over $\mathbb{R}$ as scalars and inner product $\langle\, , \rangle$, dual cone C^* of a set $C \subseteq X$ is

$$C^* = \{ y \in X : \langle y, x \rangle \geq 0 \ \forall x \in C \}$$



Specialities of finite dimensional spaces

(i) Every finite dimensional normed vector space is a Banach space

(ii) Every linear operator on a finite dimensional vector space is bounded

L. ① Examples of T

ⓐ $A: C^m \to C^n$ defined by

$Ax = b$ for $x \in C^m$ & $b \in C^n$

& $A \in C^{n \times m}$

$\|A\|$ was discussed in the case of matrices and its form depends on the norm employed in C^m

- If $A^* = A^{-1}$ then A is called orthogonal
- C^m is isomorphic to C^n if $m = n$

ⓑ $I: X \to X$ is the identity operator and is bounded for any normed space X

ⓒ Let $D: C^\infty([0,1]) \to C^\infty([0,1])$ be the differentiation linear operator on normed space of functions with

operator norm? Assume p-norm

continuous derivatives of all orders

$C^\infty([0,1])$ is normed but NOT banach

$$Du = u' \quad \forall\, u \in C^\infty([0,1])$$

(d) $T: X \to X$ where $X = \{f: C^\infty \to C^\infty\}$

and $T = \frac{d^2}{dx^2} - \frac{d}{x}$

$f_k(x) = e^{kx}$ is an eigenfunction and k^2-k the corresponding eigenvalue

(e) $T: X \to X$ where $X = \{f: \mathbb{R} \to \mathbb{R}\}$

and $T = \frac{d}{dx}$

$f_\lambda(x) = e^{\lambda x}$ is an eigenfunction and λ the corresponding eigenvalue

(f) $T: C([0,1]) \to C([0,1])$, C being space of cts fns in $[0,1]$ (with $\| \|_\infty$)

T is called the Voltera integral operator if

$$Tf(x) = \int_0^x f(y)\,dy$$ $\Big\}$ $\|T\|$ with $\|\ \|_\infty$ on $C([0,1])$

Also, T is bounded

(9) $T_R : \ell^\infty(S) \to \ell^\infty(S)$ & $T_L : \ell^\infty(S) \to \ell^\infty(S)$

st $T_R(x_1, x_2, x_3, \ldots) = \underbrace{(0, x_1, x_2, x_3 \ldots)}_{\text{Right shift operator}}$

OR $T_L(x_1, x_2, x_3, \ldots) = \underbrace{(x_2, x_3, \ldots)}_{\text{Left shift operator}}$

$\|T_R\| = \|T_L\| = 1$

$\ker(T_R) = \{0\}$ $\quad$ $\text{range}(T_R) = \{(0, x_1, \ldots) \in \ell^\infty(S)\}$

$\ker(T_L) = \{(x_1, 0, \ldots)\}$ $\quad$ $\text{range}(T_L) = \ell^\infty(S)$

(h) $T_n : L^2(D) \to \mathbb{C}$ s.t (D is [a,b] for eg)

$$T_n(f) = \frac{1}{\sqrt{2\pi}} \int_{x \in D} f(x)\, e^{-inx}\, dx$$

maps function f to its n^{th} Fourier coefficient

Then T_n is bounded $(\|T_n\| = 1)$

Q: span $(T_1(f), T_2(f), \ldots T_n(f) \ldots)$

(g) $T_y : \underbrace{X}_{\text{eg: } L^2(D)} \to \mathbb{C}$ s.t X & Y are Hilbert spaces and $y \in Y$

n

$T_y(x) = \underbrace{\langle y, x \rangle}$ is bounded and

Assume X & Y are subsets of same inner prod space

$\underbrace{\|T_y\| = \|y\|}_{\text{By Cauchy Schwarz.}}$

Eg: inner prod space $= (\mathbb{R}, *)$

$X = [a,b]$ $Y = [c,d]$