(e) If X is normed v.s & Y is Banach then $T: X \to Y$ is Banach w.r.t the operator norm

(f) $T: X \to \mathbb{R}$ is called a linear functional

Then dual of X is set of all its linear functionals

Algebraic dual $= \{T \mid T: X \to \mathbb{R}\} = X^{\#}$

If X is finite dimensional, its dual $X^{\#}$ is linearly isomorphic to X

ie if $\{e_1 \dots e_n\}$ is basis for X

then $\{g_i : X \to \mathbb{R}\}$ s.t

$$g_i\left(\sum_{j=1}^{n} x_j e_j\right) = x_i$$

form the basis for $X^{\#}$ so that for any $g \in X^{\#}$ s.t $g\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} g(e_i) x_i$

$\{g_1, g_2 \dots g_n\}$ is called the dual basis

9 Topological dual $= \{T \mid T : X \to \mathbb{R}\} = X^*$

In finite dimensional case: $X^* = X^\#$ & X^* is isomorphic to X

T is a continuous linear functional

You get specific duals for subsets of vector spaces (such as convex sets, cones and affine sets) by putting restrictions on T.

Eg: If $C \subseteq X$ s.t X is a vector space

$\boxed{\text{a) } C^\# = \text{algebraic dual cone} = \{T \in X^\# \mid T(x) \geq 0 \ \forall x \in C\}}$

in book represented as $\langle T, x \rangle$

b) Further if X is a topological vector space & $C \subseteq X$

then

$C^* =$ topological dual cone

$= \{ T \in X^* \mid T(x) \geq 0 \quad \forall x \in C \}$

Claims:

① C^* is always a convex cone
(irrespective of whether C is convex or cone or neither)

if $T_1 \in C^*$ & $T_2 \in C^*$ & $\theta_1, \theta_2 \geq 0$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \in X^*$$

$$\theta_1 T_1(x) + \theta_2 T_2(x) \geq 0$$

$\Rightarrow C^*$ is a convex cone

(Similarly $C^\#$ is also always a convex cone)

(2) If X is finite dimensional,

$$C^{\#} = C^{*}$$

since $X^{\#} = X^{*}$

(3) If X is a Hilbert space,

C^{*} is closed .... More properties follow when X is a Hilbert space

Specialities of finite dimensional spaces

(i) Every finite dimensional normed vector space is a Banach space

(ii) Every linear operator on a finite dimensional vector space is bounded/continuous

. - . - H/W: Complete

h) Riesz representation theorem:

As such we are looking at topological dual so T is cts/bnded

If $T: X \to \mathbb{R}$ and X is Hilbert and T is bounded, then $\exists$ a unique vector $y \in X$ s.t

$$T(x) = \langle y, x \rangle \quad \forall\, x \in X$$

$$X^* = \{ T_y(x) = \langle y, x \rangle \mid x \in X \}$$

is the dual of X^*

Defines a linear functional in terms of an inner product

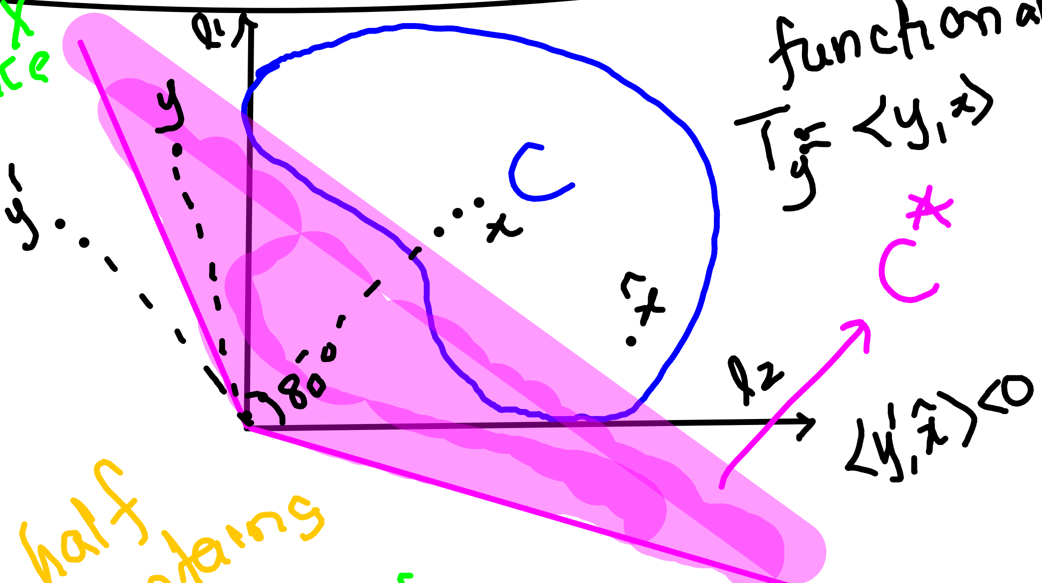
Further, X & X^* are isomorphic.

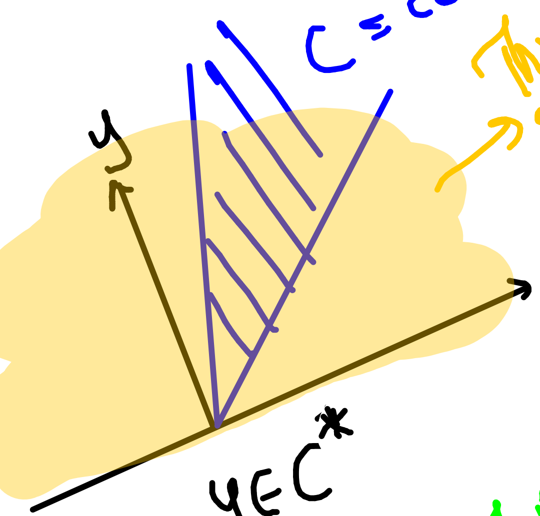
(i) Thus, if X is a Hilbert space over $\mathbb{R}$ as scalars and inner product $\langle\,,\rangle$, dual cone C^* of a set $C \subseteq X$ is

Since C^* is restricted to cts linear fnls, boundedness is guaranteed

$$C^* = \{\underbrace{y}_{T\in X^* \text{ since}} \in X : \langle y, x\rangle \geq 0 \ \forall x \in C\}$$

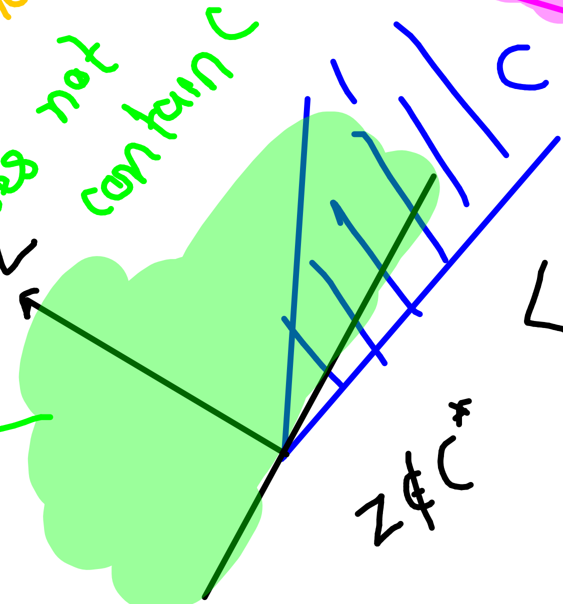
In $\mathbb{R}^n$, etc this is intersection of half spaces





This half space contains C

This halfspace does not contain C



$\langle z, x\rangle < 0$ for some $x \in C$

Properties of dual cones

① If X is a Hilbert space &

$C \subseteq X$ then C^* is a closed

convex cone

↳ We have already proved that

C^* is a convex cone

↳ C^* = intersection of infinite

closed topological half spaces

$$C^* = \bigcap_{x \in C} \{ y \mid y \in X, \langle y, x \rangle \geq 0 \}$$

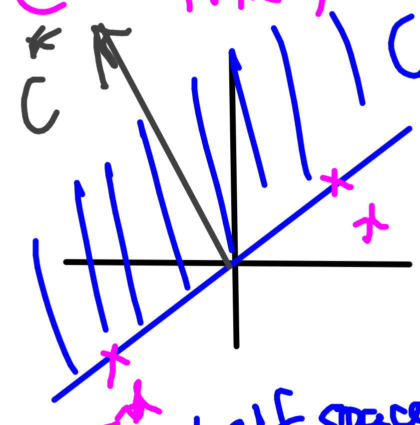
$\Rightarrow C^*$ is closed

② $C_1 \subseteq C_2 \Rightarrow C_2^* \subseteq C_1^*$

③ $\text{interior}(C^*) = \{ y \in X \mid \langle y, x \rangle > 0 \ \forall x \in X^? \}$

④ If C is a cone and has $\text{int}(C) \neq \emptyset$ the C^* is pointed

↳ i.e if $x \in C^*$ & $-x \in C^*$ then $x = 0$



⑤ If C is a cone then $\text{closure}(C) = C^{**}$

if C = open half space, C^{**} = closed half space

⑥ If closure of C is pointed then $\text{interior}(C^*) \neq \emptyset$

s is called conically spanning set of cone K iff $\text{conic}(s) = K$

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

Easy to prove it is a cone

$\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Q: Is it convex? ✓

Is it a cone? ✗

Since $0 \notin S^n_{++}$

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$