Properties of dual cones

① If X is a Hilbert space & $C \subseteq X$ then C^* is a closed convex cone

↳ We have already proved that C^* is a convex cone

↳ C^* = intersection of infinite closed topological half spaces

$$C^* = \bigcap_{x \in C} \{y \mid y \in X, \langle y, x \rangle \geq 0\}$$

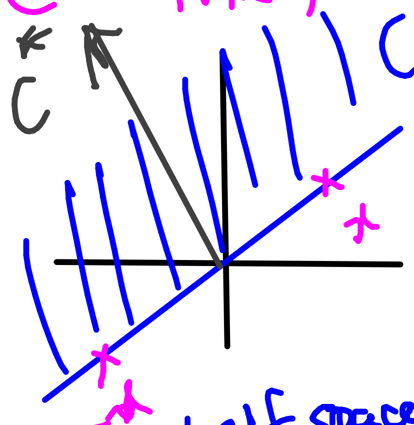
$\Rightarrow C^*$ is closed

② $C_1 \subseteq C_2 \Rightarrow C_2^* \subseteq C_1^*$

③ $\text{interior}(C^*) = \{y \in X \mid \langle y, x \rangle > 0 \ \forall x \in X\}$

④ If C is a cone and has $\text{int}(C) \neq \emptyset$ the C^* is pointed

↳ i.e if $x \in C^*$ & $-x \in C^*$ then $x = 0$



⑤ If C is a cone then $\text{closure}(C) = C^{**}$

If C = open half space, C^{**} = closed half space

⑥ If closure of C is pointed then $\text{interior}(C^*) \neq \emptyset$

S is called conically spanning set of cone K iff $\text{conic}(S) = K$

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

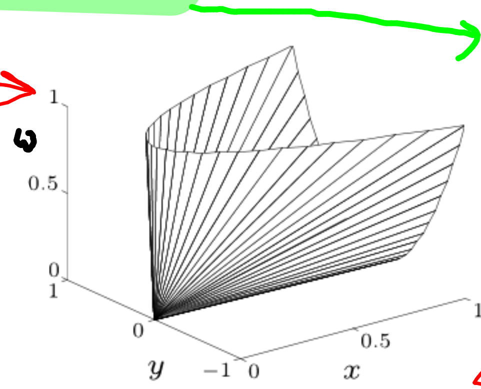
Easy to prove it is a cone

$\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Q: Is it convex? ✓
Is it a cone? ✗
Since $0 \notin S^n_{++}$

example: $\begin{bmatrix} x & y \\ y & w \end{bmatrix} \in \mathbf{S}^2_+$



$S^2_+ \ldots \mathbb{R}^3$

Notes abt p.d cone: (or psd cone)

$$S_+^n = \{A \in S^n \mid A \succeq 0\} = \{A \in S^n \mid y^TAy \geq 0 \ \forall \|y\|=1\}$$

$$= \bigcap_{\|y\|=1} \{A \in S \mid \langle y^Ty, A\rangle \geq 0\}$$

$$y^TAy = \sum_j \sum_i y_i a_{ij} y_j = \sum_i \sum_j (y_i y_j) a_{ij}$$

$$= \langle yy^T, A\rangle = \mathrm{tr}\left((yy^T)^TA\right) = \mathrm{tr}(yy^TA)$$

= Intersection of infinite # of half spaces belonging to $\mathbb{R}^{n(n+1)/2}$ [Dual representation]

$y = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \Rightarrow yy^T = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$

H/W: Plot a finite # of halfspaces parametrized by θ

Cone boundary consists of all singular p.s.d matrices having atleast one 0 eigenvalue

H/W

Interior consists of all full rank matrices A (rank $A = m$) ie $A \succ 0$

ORIGIN = 0 = matrix with all 0 eigenvalues

Claim: $(S_+^n)^* = S_+^n$

ie $\langle X, Y\rangle = \mathrm{tr}(X^TY) = \mathrm{tr}(XY) \geq 0 \ \forall X \in S_+^n$

$\Updownarrow$ iff

$Y \in S_+^n$

Proof: (a) Let us say $Y \notin S_+^n$. That is $\exists\, z \in \mathbb{R}^n$ s.t

$$z^T Y z = tr(\underbrace{zz^T}_{X} Y) < 0$$

i.e $\exists\, X = zz^T \in S_+^n$ s.t $\langle X, Y\rangle < 0$

$$\boxed{\Rightarrow Y \notin (S_+^n)^*}$$

(b) Suppose $Y, X \in S_+^n$. Any $X \in S_+^n$ can be written in terms of eigenvalue decomposition as

$$X = \sum_{i=1}^n \lambda_i u_i u_i^T \quad (\lambda_i \geq 0)$$

$$\therefore \langle Y, X\rangle = tr(YX) = tr\left(Y \sum_{i=1}^n \lambda_i u_i u_i^T\right)$$

$$= \sum_{i=1}^n \lambda_i tr(Y u_i u_i^T)$$

$$= \sum_{i=1}^n \underbrace{\lambda_i}_{\geq 0 \text{ since } X \in S_+^n} \underbrace{u_i^T Y u_i}_{\geq 0 \text{ since } Y \in S_+^n} \geq 0$$

$$\boxed{\Rightarrow Y \in (S_+^n)^*}$$

Q: Is there some connection between $Y = yy^T$ used for $S_+^n = \bigcap_{\|y\|=1} \{X \in S^n \mid \langle yy^T, X\rangle \geq 0\}$ $\|y\| = 1$ and $(S_+^n)^* = S_+^n$

(To be revisited as H/W)

Q: $(S^n_{++})^* = ?$ $\quad \text{int}(S^n_+) = S^n_{++}$

Ans: $(S^n_{++})^* = S^n_+$ (will be done formally for general case of convex cones)

C = convex cone

$C^{**} = \text{cl}(C)$

Q: Consider an application of psd cone for optimization. (thru LP)

↳ We will first see (weak) duality in a linear optimization problem (LP)

↳ Next we look at generalized (conic) inequalities and properties that the cone must satisfy for the inequality to be a valid inequality

↳ Next we generalize LP to conic program (CP) using generalized inequality and realize weak duality for CP thru dual cones

Consider

linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

LP

Affine objective

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\leq 0 \end{array}$$

Conic Program (CP)

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\leq_K 0 \end{array}$$

Let: $\lambda \geq 0$ (i.e $\lambda \in \mathbb{R}^n_+$)

Then $\lambda^T(-Ax+b) \leq 0$

$$\Rightarrow c^Tx \geq c^Tx + \lambda^T(-Ax+b)$$

$$= \lambda^T b + (c - A^T\lambda)^T x$$

$$\geq \min_x \lambda^T b + (c - A^T\lambda)^T x$$

$$= \begin{cases} \lambda^T b & \text{if } A^T\lambda = c \\ -\infty & \text{if } A^T\lambda \neq c \end{cases}$$

independent of λ

independent of x

$\Rightarrow$

$$\boxed{\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^Tx \\ \text{s.t} & Ax \geq b \end{array} \quad \geq \quad \begin{array}{ll} \max\limits_{\lambda \geq 0} & b^T\lambda \\ \text{s.t} & A^T\lambda = c \end{array}}$$

Primal LP (lower bounded)

Dual LP (upper bounded)

We will motivate through linear programming (LP) generalised inequalities:

$$\min_{x \in \mathbb{R}^n} \; c^T x \quad \text{subject to } -Ax+b \le 0$$

LINEAR PROGRAM

Can be rewritten as $Ax \geq b$ or $Ax - b \in \mathbb{R}^n_+$

Note: $\mathbb{R}^n_+$ is a CONE. How abt defining generalised inequality for a cone K as: $c \geq_K d$ iff $c - d \in K$ and a general conic program as:

$$\min_{x \in \mathbb{R}^n} \; c^T x \quad \text{subject to } -Ax+b \le_K 0$$

CONIC PROGRAM

That is, $Ax - b \in K$

Generalized inequalities

Also referred to as a regular cone

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Some restrictions on K that we will require. H/w: WHY?

i.e. if $a, -a \in K$, then $a = 0$

i.e. K has no st. lines passing thru 0

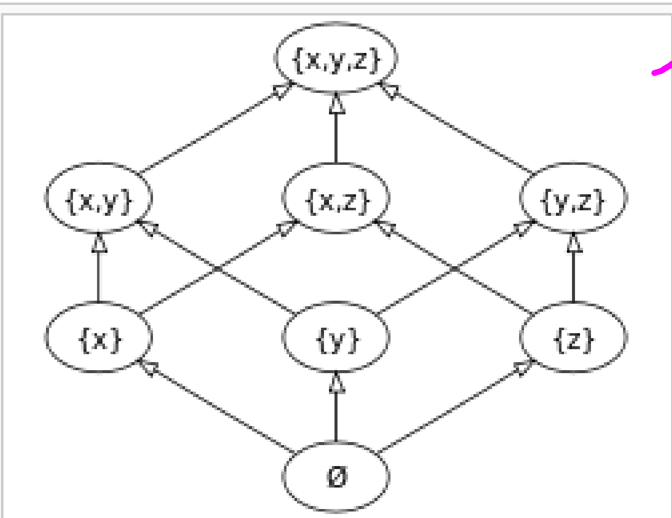
examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

To prove that K being closed, solid & pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, recall that any partial order $\geq$ should satisfy the following properties (refer page 51 of www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf)

1. *Reflexivity:* $a \geq a$;
2. *Anti-symmetry:* if both $a \geq b$ and $b \geq a$, then $a = b$;
3. *Transitivity:* if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. *Compatibility with linear operations:*
   (a) *Homogeneity:* if $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$
   ("One can multiply both sides of an inequality by a nonnegative real")
   (b) *Additivity:* if both $a \geq b$ and $c \geq d$, then $a + c \geq b + d$
   ("One can add two inequalities of the same sign").



The Hasse diagram of the set of all subsets of a three-element set {x, y, z}, ordered by inclusion.

→ example partial order $\subseteq$ over sets

(source: http://en.wikipedia.org/wiki/Partially_ordered_set)

that is, the $\subseteq$ partial order