Let us resume our discussion on linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

LP

Affine objective

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\le 0 \end{array}$$

Let: $\lambda \geq 0$ (i.e $\lambda \in \mathbb{R}^n_+$)

Then $\lambda^T(-Ax+b) \leq 0$

$\Rightarrow c^Tx \geq c^Tx + \lambda^T(-Ax+b)$

$= \lambda^Tb + (c - A^T\lambda)^T x$

$\geq \min\limits_x \lambda^Tb + (c - A^T\lambda)^T x$

$= \begin{cases} \lambda^Tb & \text{if } A^T\lambda = c \\ -\infty & \text{if } A^T\lambda \neq c \end{cases}$

independent of λ

independent of x

$\Rightarrow$

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^Tx \\ \text{s.t} & Ax \geq b \end{array} \quad \geq \quad \begin{array}{ll} \max & b^T\lambda \\ \lambda \geq 0 & \\ \text{s.t} & A^T\lambda = c \end{array}$$

Primal LP (lower bounded)

Dual LP (upper bounded)

Conic Program (CP)

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\le_K 0 \end{array}$$

K is a regular/proper cone

Generalised cone program

$$\begin{array}{ll} \min\limits_{x\in S} & \langle c, x\rangle_S \\ \text{subject to} & Ax - b \in K \end{array}$$

We need an equivalent $\lambda \in K^*$ s.t

$\langle \lambda, Ax-b \rangle \geq 0$

This K^* s.t

$K^* = \{\lambda \mid \langle \lambda, Ax-b\rangle \geq 0 \ \forall Ax-b \in K\}$

is called the DUAL CONE of K (ie element that have +ve inner prod with each element of K) by dual)

by primal)

Called the weak duality theorem for Linear Program

$K_* = \{\lambda : \lambda^T \xi \geq 0 \forall \xi \in K\}$ is the cone dual to K

{defn on page 7 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf}

With this, prove the following weak duality theorem for CONIC PROGRAM

$$\min_{x \in S} \langle c, x \rangle \text{ s.t. } Ax \geq_K b \quad \geq \quad \max_{\lambda \in K^*} \langle b, \lambda \rangle \text{ s.t. } A^T\lambda = c$$

Primal CP (lower bounded by dual)

Dual CP (upperbounded by primal)

Note: ① Both LP & CP dealt with affine objective

② CP dealt with the generalised conic inequalities

③ Later, in convex programs, we will deal with the more general convex functions in the objective

Notes:

① If $K = \mathbb{R}^n_+$, the CP is an LP

If $K = S^n_+$, the CP is an SDP

S^n_+: Set of all $n \times n$ symmetric positive semidefinite matrices

SDP: Semi-definite program

② Any generic convex program can be expressed as a cone program (CP) [H/W]

HOW ABOUT STRONG DUALITY FOR LPs?

[Page 21 of http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf]

Theorem 1.2.2 [Duality Theorem in Linear Programming] *Consider a linear programming program*

$$\min_x \left\{ c^T x \,\middle|\, Ax \geq b \right\} \tag{LP}$$

along with its dual

$$\max_y \left\{ b^T y \,\middle|\, A^T y = c, y \geq 0 \right\} \tag{LP*}$$

Then

1) *The duality is symmetric: the problem dual to dual is equivalent to the primal;*

2) *The value of the dual objective at every dual feasible solution is $\leq$ the value of the primal objective at every primal feasible solution*

Weak LP duality (already proved)

3) *The following 5 properties are equivalent to each other:*

(i) *The primal is feasible and bounded below.* $\exists x$ s.t. $Ax \geq b$; $-\infty < c^T x$, $Ax \geq b$

(ii) *The dual is feasible and bounded above.* $b^T y < \infty$, $A^T y = c$

(iii) *The primal is solvable.* $\exists \lambda$ s.t. $A^T \lambda = c$; LP has a soln

(iv) *The dual is solvable.* LP* has a soln

(v) *Both primal and dual are feasible.*

Whenever (i) $\equiv$ (ii) $\equiv$ (iii) $\equiv$ (iv) $\equiv$ (v) is the case, the optimal values of the primal and the dual problems are equal to each other.: Strong duality = (2) + (3)

Proof of (1) from page 21 of http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf

Proof. 1) is quite straightforward: writing the dual problem (LP*) in our standard form, we get

$$\min_y \left\{ -b^T y \,\middle|\, \begin{bmatrix} I_m \\ A^T \\ -A^T \end{bmatrix} y - \begin{pmatrix} 0 \\ -c \\ c \end{pmatrix} \geq 0 \right\},$$

where I_m is the m-dimensional unit matrix. Applying the duality transformation to the latter problem, we come to the problem

$$\max_{\xi,\eta,\zeta} \left\{ 0^T \xi + c^T \eta + (-c)^T \zeta \,\middle|\, \begin{array}{rcl} \xi & \geq & 0 \\ \eta & \geq & 0 \\ \zeta & \geq & 0 \\ \xi - A\eta + A\zeta & = & -b \end{array} \right\},$$

which is clearly equivalent to (LP) (set $x = \eta - \zeta$).

Proof for (i) ≡ (ii) ≡ (iii) ≡ (iv) ≡ (v).

Let us prove that (i) ⇒ (iv)

Let l be a lower bound on the primal. That is, $\boxed{\forall x \text{ s.t. } Ax \geq b, \; c^T x \geq l}$

We will first prove that:

$\boxed{\text{Recall: (1) } Ax = b \text{ has a soln iff } y^T b = 0 \text{ for all } y \text{ s.t. } y^T A = 0}$ [can be proved by analyzing Gauss Elimination]

called Theorem of Alternate (Results in Farkas Lemma)

(2) Exactly one of the following statements are true:

① $\exists x \in \mathbb{R}^n$ s.t. $Ax = b$, $x \geq 0$

② $\exists y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ $y^T b < 0$

Independent proof of Farkas' lemma:

(a) NOT BOTH Suppose x satisfies (1) & y satisfies (2). Then $0 > y^T b \geq y^T A x \geq 0$

$\because Ax = b$ $\quad \because y^T A \geq 0$ & $x \geq 0$

$\Rightarrow$ CONTRADICTION

(b) At least one Suppose (1) is infeasible

We will show that (2) is feasible

(i) Consider the closed convex set

$$S = \{Ax : x \geq 0\}$$

Then $b \notin S$ since (1) is infeasible

(ii) According to the separating hyperplane theorem, $\exists$ a hyperplane separating S from $\{b\}$. That is, $\exists$ $y \in \mathbb{R}^m$ & $l \in \mathbb{R}$ s.t. $y^T b < l$ & $y^T s \geq l$ $\forall s \in S$

(iii) Since $0 \in S$, $l \leq 0 \Rightarrow y^T b < 0$

(iv) $s = Ax$ for some $x \geq 0$

$\therefore y^T A x \geq l$ $\forall x \geq 0 \Rightarrow y^T A \geq 0$

LHS should hold for arbitrarily large $x \geq 0$

Farkas' lemma corollary

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, exactly one of the following two systems holds:

① $\exists x \in \mathbb{R}^n$ s.t $Ax \geq b$.

② $\exists y \in \mathbb{R}^m$ s.t $y^T A = 0$ & $y^T b > 0$

Proof: Apply Farkas' lemma to:

① $\exists x \in \mathbb{R}^n, s \in \mathbb{R}^m$ s.t

$$Ax - Is = b, \quad x, s \geq 0$$

② H/W: Fill it up

we know that

whenever $Ax \geq b$, $c^T x \geq \ell$

ie the system below has no soln:

$-c^T x \geq -\ell$, $Ax \geq b$

$$\boxed{\text{3} \quad \exists\, y \text{ s.t } A^T y = c \ \& \ b^T y \geq \ell}$$

Apply Farkas' corollary & formulate 2 problems, exactly one of which has a solution

1. $\exists\, x \in \mathbb{R}^n$ s.t $Ax \geq b$
   $-c^T x \geq -\ell$

The lower bound on primal as per (i)

2. $\exists\, y \in \mathbb{R}^m$ s.t $A^T y = c$
   $y^T b \geq \ell$
   $y \geq 0$

Theorem 1.2.3 [Necessary and sufficient optimality conditions in linear programming] *Consider an LP program* (LP) *along with its dual* (LP*). *A pair* (x, y) *of primal and dual feasible solutions is comprised of optimal solutions to the respective problems if and only if*

$$y_i[Ax - b]_i = 0,\ i = 1, ..., m, \qquad \text{[complementary slackness]}$$

likewise as if and only if

$$c^T x - b^T y = 0 \qquad \text{[zero duality gap]}$$

special case of Karush Kuhn Tucker (KKT) conditions to be discussed later

Proof sketch: [H/W Complete the proof rigorously]

$\boxed{\text{Only if}}$ From Theorem 1.2.2, if x & y are pts of optimal primal & dual solns respectively, then

$$A^T y = c \Rightarrow (A^T y)^T x = c^T x = y^T b \Rightarrow y^T(Ax - b) = 0$$

(and since $y \geq 0$, $Ax - b \geq 0$)

$$\Rightarrow \forall i,\ y_i[Ax - b]_i = 0$$

$\boxed{\text{if}}$ $y_i[Ax-b]_i = 0 \Rightarrow y^T(Ax - b) = 0 \Rightarrow y^T b = y^T A x$

$\Rightarrow$ Dual is solvable (condition 3.(iv) of Theorem 1.2.2)

can be proved independently

$\Rightarrow$ Conditions (1) & (3) of Theorem 1.2.2 are met

Similar Duality theorem for CP:

[page 7 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

Note dual(dual(K)) = K if K is a closed

dual(K) is ALWAYS a closed cone

$$\min_x \left\{ c^T x : \underbrace{Ax - b \geq_K 0}_{\Leftrightarrow Ax - b \in K} \right\} \qquad \text{(CP)}$$

$$\max_\lambda \left\{ b^T \lambda : A^T \lambda = c, \lambda \geq_{K_*} 0 \right\}, \qquad \text{(D)}$$

Theorem 2.1. *Assuming A in* (CP) *is of full column rank, the following is true:*

(i) *The duality is symmetric:* (D) *is a conic problem, and the conic dual to* (D) *is (equivalent to)* (CP);

(ii) [weak duality] Opt(D) $\leq$ Opt(CP); → Already proved

(iii) [strong duality] *If one of the programs* (CP), (D) *is bounded and strictly feasible* (i.e., the corresponding affine plane intersects the interior of the associated cone), *then the other is solvable and* Opt(CP) = Opt(D). *If both* (CP), (D) *are strictly feasible, then both programs are solvable and* Opt(CP) = Opt(D);

(iv) [optimality conditions] *Assume that both* (CP), (D) *are strictly feasible. Then a pair* (x, λ) *of feasible solutions to the problem is comprised of optimal solutions iff* $c^T x = b^T \lambda$ *("zero duality gap"), same as iff* $\lambda^T[Ax - b] = 0$ *("complementary slackness").*

H/W: Prove these in a manner similar to the duality theorem for LP

Note: Duality theorems for LP and CP are special cases of Lagrange duality that we will discuss later in the course

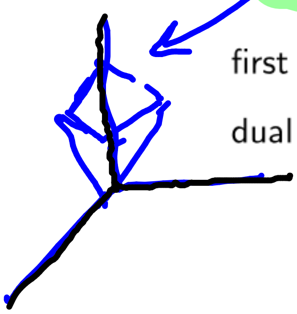
Dual cones and generalized inequalities

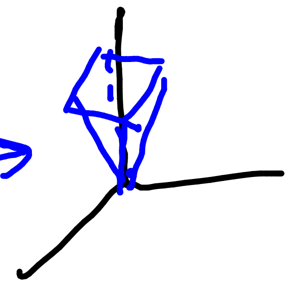
dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$





first three examples are **self-dual** cones

dual cones of proper cones are proper, hence define generalized inequalities:

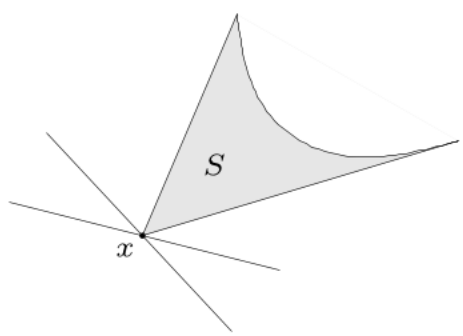
$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

if $\frac{1}{p} + \frac{1}{q} = 1$ then $\|x\|_p$ is called the dual norm of $\|x\|_q$

Minimum and minimal elements via dual inequalities

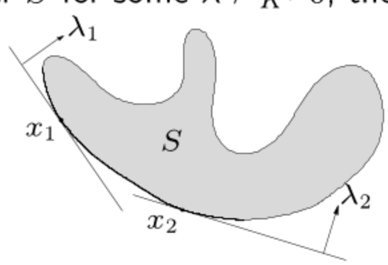
minimum element w.r.t. $\preceq_K$

x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. $\preceq_K$

- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal



- if x is a minimal element of a *convex* set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

Thus (continuing our story of dual descriptions of sets)

3) if C = Conic set $\subseteq S$ and B is its basis

$$C = \text{conichull}(B) = \{s \in S \mid \langle s, b\rangle_S \geq 0 \ \forall\, b \in B^*\}$$

where B^* is basis for $C^* = \{s^* \in S \mid \langle s^*, c\rangle_S \geq 0 \ \forall\, c \in C\}$

Some more facts on dual cones:

ⓐ Dual cone $K^* = \{y \mid \langle x, y\rangle \geq 0 \ \ \forall\, x \in K\}$ is convex, even if K is not,

i.e, a cone   i.e, a cone

Proof: (i) $0 \in K^*$ since $\langle x, 0\rangle = 0$

(ii) if $y_1 \in K^*$ & $y_2 \in K^*$ and $\underbrace{\theta_1, \theta_2 \geq 0}_{\text{Reals}}$

then $\langle x, \theta_1 y_1 + \theta_2 y_2\rangle = \bar{\theta}_1 \langle x, y_1\rangle + \bar{\theta}_2 \langle x, y_2\rangle$

$$= \theta_1 \langle x, y_1\rangle + \theta_2 \langle x, y_2\rangle \geq 0$$

In fact K^* is a closed convex cone

ⓑ If $K_1 \subseteq K_2 \Rightarrow$

(c) Interior(K^*) $= \{y \mid y^T x > 0 \ \forall \ x \in K\}$

(d) if interior(K) $\neq \emptyset$, K^* is pointed

(e) K^{**} is the closure of K [$\therefore$ if K is closed, $K^{**} = K$). Also if K is a proper cone, so is its dual

(f) if closure of K is pointed then K^* has non empty interior [Refer Boyd's book, exercise 2.31 for (a) - (f)

(g) if $V \subseteq \mathbb{R}^n$ is a subspace, its dual cone is its orthogonal complement

a subspace is a cone

$$V^* = V^\perp = \{y \mid y^T v = 0 \ \forall \ v \in V\}$$

h) FROM DUAL OF NORM CONE TO DUAL NORM

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$

The dual of $K = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \le t\}$

is $K^* = \{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \le v\}$

Where $\|u\|_* = \sup\{u^T x \mid \|x\| \le 1\}$

Proof: We need to show that

$$x^T u + tv \ge 0 \text{ whenever } \|x\| \le t \iff \|u\|_* \le v. \tag{2.20}$$

1) Let us start by showing that the righthand condition on (u, v) implies the lefthand condition. Suppose $\|u\|_* \le v$, and $\|x\| \le t$ for some $t > 0$. (If $t = 0$, x must be zero, so obviously $u^T x + vt \ge 0$.) Applying the definition of the dual norm, and the fact that $\|-x/t\| \le 1$, we have

$$u^T(-x/t) \le \|u\|_* \le v,$$

and therefore $u^T x + vt \ge 0$.

2) Next we show that the lefthand condition in (2.20) implies the righthand condition in (2.20). Suppose $\|u\|_* > v$, *i.e.*, that the righthand condition does not hold. Then by the definition of the dual norm, there exists an x with $\|x\| \le 1$ and $x^T u > v$. Taking $t = 1$, we have

$$u^T(-x) + v < 0,$$

which contradicts the lefthand condition in (2.20).