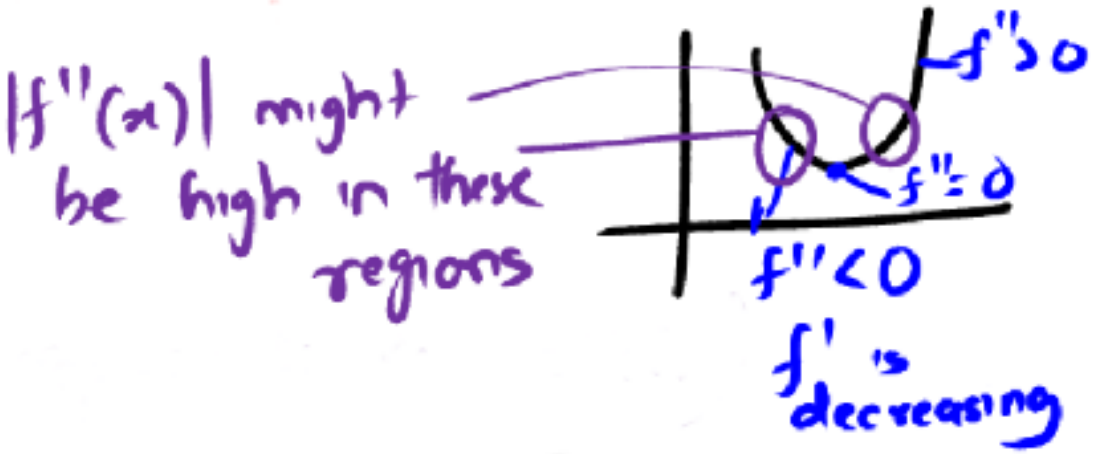
Lipschitz continuity

- Intuitively, a Lipschitz continuous function is limited in how fast its gradient (slope) can change: there exists a definite real number such that, for every pair of points on the graph of the gradient, the absolute value of the slope of the line connecting them is not greater than this real number
  - This bound is called the function's Lipschitz constant, $L > 0$
- $\left\|\nabla f(x) - \nabla f(y)\right\| \leq L\|x - y\|$

Interpretation of Lipschitz continuity

- Consider $\nabla f(x) \in \mathbf{R}$, and $\nabla f(x) = \frac{df}{dx} = f'(x)$
- $|f'(x) - f'(y)| \leq L|x - y|$
  $\implies \frac{f'(x)-f'(y)}{|x-y|} \leq L$
  $\implies \left|\frac{f'(x+h)-f'(x)}{h}\right|$ (putting $y = x + h$)
- Taking limit $h \to 0$, we get
  $|f''(x)| \leq L$
- f'' represents curvature



Example: $f(x) = \frac{x^3}{3}$

- $f(x) = \frac{x^3}{3} \implies f'(x) = x^2$
- **Claim:** $f(x)$ / $f'(x)$ is locally Lipschitz continuous but not globally
- Consider $x \in \mathbf{R}$
- $\sup_{y \in (x-1, x+1)} |f''(y)| = \sup_{y \in (x-1, x+1)} |2y| \leq 2|x| + 1$
- Applying mean value theorem:
  $\exists\, (y, z) \in (x-1, x+1)^2, \lambda$
  $f''(\lambda) = \frac{f'(y) - f'(z)}{y - z}$

$f(x) = x^3/3$

- $|f'(y) - f'(z)| = |f''(\lambda)(y - z)|$
  $\leq |2|x| + 1|\,|y - x|,\ \forall (y, z) \in (x - 1, x + 1)^2$
- Thus, $L = |2|x| + 1|$ → If $y \in (x-1, x+1)$ · $f''(y) \leq L(x)$
- Therefore, f is Lipschitz continuous in $(x - 1, x + 1)$
- But as $x \to \infty$, $L \to \infty$
- This implies that f may not be Lipschitz continuous everywhere
- Consider $y \neq 0$, and
  $\frac{f'(y) - f'(0)}{|y - 0|} = |y|$
- $|y| \to \infty$ as $y \to \infty$
- Thus, f is proved to not be Lipschitz continuous globally

Another example

- Consider

$$f(x) = \begin{cases} x^2 sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- We can verify that this function is continuous and differentiable everywhere
  i.e. $f''(0) = 0$ from left and right
- However, we can show that $f(x)$ is *not* Lipschitz continuous

Lipschitz continuity: another example

- **Consider:** $f'(x) = |x|$
- Since $|f'(x) - f'(y)| = \big||x| - |y|\big| \leq |x - y|$,
  f is Lipschitz continuous with $L = 1$
- However, it is not differentiable everywhere (not at 0)
- In fact, if f is differentiable everywhere, it is also Lipschitz continuous
- f is continuous $\supseteq$ f is differentiable $\supseteq$ f' is continuous $\supseteq$ f' is Lipschitz continuous $\supseteq$ f' is differentiable

Considering gradients in Lipschitz continuity

- If f is Lipschitz continuous, then

$$\left\|\nabla f(x) - \nabla f(y)\right\| \leq L\|x - y\|$$

- **Taylor's theorem** states that if f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous in the closed interval $[a, b]$, and differentiable in (a, b), then there exists a number $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(b-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(b-a)^{n+1}$$

- We will invoke Taylor's theorem up to the second degree:

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(c)(y-x)^2$$

where $c \in (x, y)$ and $x, y \in \mathbf{R}$

- Let us generalize to $f\colon \mathbf{R}^n \to \mathbf{R}$:

$$f(y) = f(x) + \nabla^\top f(x)(y-x) + \frac{1}{2}(y-x)^\top \nabla^2 f(c)(y-x)$$

where $c = x + \Gamma(y-x)$, $\Gamma \in (0,1)$, and $x, y \in \mathbf{R}^n$

For a Lipschitz continuous $f: \mathbf{R}^n \to \mathbf{R}$, we can show that for any vector v,

- $v^\top \nabla^2 f(x) v \leq v^\top L v$
  $\implies v^\top (\nabla^2 f(x) - LI) v \leq 0$
- That is, $\nabla^2 f(x) - LI$ is negative semi-definite
- This can be written as:

$$\nabla^2 f(x) \preceq LI$$

- Convexity:

$$f(y) \geq f(x) + \nabla^\top f(x)(y - x)$$

- Strict convexity:

$$f(y) > f(x) + \nabla^\top f(x)(y - x)$$

- Strong convexity:

$$f(y) \geq f(x) + \nabla^\top f(x)(y - x) + \frac{m}{2}\|y - x\|^2$$

  - Strong convexity implies strict convexity
  - $\frac{m}{2}\|y - x\|^2$ can be 0 only when $y = x$