Q1: Define the sum of two sets S_1 & S_2 as:

$$S_1 + S_2 = \{x+y \mid x \in S_1, y \in S_2\}.$$

Also consider the partial sum of $S_1, S_2 \in R^n \times R^m$ as:

$$\{(x, y_1+y_2) \mid (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

where $x \in R^n$ & $y_1, y_2 \in R^m$.

Show (using properties of convex sets) that:

If S_1 & S_2 are convex, their $\underbrace{\text{sums}}_{(a)}$ & $\underbrace{\text{partial sums}}_{(b)}$ are also convex.

Similarly, show that the $\underbrace{\text{projection}}_{(c)}$ of a convex set S onto some of its coordinates is convex

i.e if $S \subseteq S_1 \times S_2$ is convex then

$$T = \{x_1 \in S_1 \mid (x_1, x_2) \in S \text{ for some } x_2 \in S_2\}$$

is convex

Hint: (a) Image under affine f (b) Image under affine f (c) Image under affine f

$A \preceq B$ iff $B-A$ is psd

Q2: The condition

$$A(x) = x_1 A_1 + x_2 A_2 \cdots + x_n A_n \preceq B$$

where $B, A_i \in S^m$ is called a LINEAR MATRIX INEQUALITY (LMI) in x. Prove that the solution set of LMI: $\{x \mid A(x) \preceq B\}$ is convex using the fact that the PSD cone is convex.

<u>Hint:</u> Inverse image of psd cone under affine function

Q1: Prove that unit balls of norms on $\mathbb{R}^n$ are exactly the same as convex sets V in $\mathbb{R}^n$ satisfying the following 3 properties:

(a) V is symmetric w.r.t origin: $x \in V \Rightarrow -x \in V$

(b) V is bounded & closed

(c) V contains a neighborhood of the origin

A set V satisfying the outlined properties is the unit ball of the norm

$$\|x\| = \inf\{t \geq 0 : t^{-1}x \in V\}$$

[HINT: You could exploit the following facts]

(1) A norm $\|\cdot\|$ on $\mathbb{R}^n$ is Lipschitz continuous with respect to the standard Euclidean distance: $\exists\, C_{\|\cdot\|} < \infty$ s.t $\|x\| - \|y\| \leq C_{\|\cdot\|} \|x-y\|_2$ for all x, y

(2) Vice versa, the Euclidean norm is Lipschitz continuous with respect to a given norm $\|\cdot\|$: there exists $c_{\|\cdot\|} < \infty$ s.t $\|x\|_2 - \|y\|_2 \leq c_{\|\cdot\|} \|x-y\|$ $\forall x, y$