# Homework Exercise 3

**Due on** $12^{th}$ **September, 2009**

1. Find and classify (as local or global maximum or minimum or as a saddle point) the stationary points for the following function. Solve using both (computerized) plots as well as analytic method to confirm your solution. Both will be graded.

$$f(x) = 2x_1^2 + x_2^2 - 2x^1x^2 + 2x_1^3 + x_1^4$$

   (**1 Mark for analytically solving and 1 Mark for illustrating through plot**)

2. Let $f(\mathbf{x})$ defined on a domain $\mathcal{D} \subseteq \Re^n$ have a local maximum or minimum at $\mathbf{x}^*$ and let the first-order partial derivatives exist at $\mathbf{x}^*$. Consider the function

$$g_i(x_i) = f(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_n^*)$$

   Prove that, if $f$ has a local extremum at $\mathbf{x}^*$, then each function $g_i(x_i)$ must have a local extremum at $x_i^*$.

   (**1 Mark**)

3. Let $f : \mathcal{D} \rightarrow \Re$ where $\mathcal{D} \subseteq \Re^n$. Let $f(\mathbf{x})$ have continuous partial derivatives and continuous mixed partial derivatives in an open ball $\mathcal{R}$ containing a point $\mathbf{x}^*$ where $\nabla f(\mathbf{x}^*) = 0$. Let $\nabla^2 f(\mathbf{x})$ denote an $n \times n$ matrix of mixed partial derivatives of $f$ evaluated at the point $\mathbf{x}$, such that the $ij^{th}$ entry of the matrix is $f_{x_i x_j}$. The matrix $\nabla^2 f(\mathbf{x})$ is called the Hessian matrix. The Hessian matrix is symmetric[1].

   Prove/disprove that if $\nabla^2 f(\mathbf{x}^*)$ is positive definite, *i.e.*, $\nabla^2 f(\mathbf{x}^*) \succ 0$, there exists an $\epsilon > 0$, with $\mathcal{B}(\mathbf{x}^*, \epsilon) \subseteq \mathcal{R}$ such that for all $||\mathbf{h}|| < \epsilon$, $\nabla^2 f(\mathbf{x}^* + \mathbf{h}) \succ 0$.

   (**2.5 Marks**)

[1] By Clairauts Theorem, if the partial and mixed derivatives of a function are continuous on an open region containing a point $\mathbf{x}^*$, then $f_{x_i x_j}(\mathbf{x}^*) = f_{x_j x_i}(\mathbf{x}^*)$, for all $i, j \in [1, n]$.