References

- Convex Optimization by Stephen Boyd and Lieven Vandenberghe
- Lectures on Modern Convex Optimization by Aharon Ben-Tal and Arkadi Nemirovski
- Convex Analysis by R. T. Rockafellar, Vol. 28 of Princeton Math. Series, Princeton Univ. Press, 1970 (470 pages)
- Numerical Optimization by Nocedal, Jorge, Wright, Stephen
- Introduction to Nonlinear Optimization - Theory, Algorithms and Applications by Amir Beck

Developing Tools for Convexity Analysis of $f(x_1, x_2, ..x_n)$

Instructor: Prof. Ganesh Ramakrishnan

Summary of Optimization Principles for Univariate Functions

Detailed slides at https://www.cse.iitb.ac.in/~cs709/notes/enotes/2-08-01-2018-univariateprinciples.pdf, video at https://tinyurl.com/yc4d2aqg and Section 4.1.1 (pages 213 to 214) of the notes at https://www.cse.iitb.ac.in/~cs709/notes/BasicsOfConvexOptimization.pdf.

Maximum and Minimum values of univariate functions

Let $f : \mathcal{D} \rightarrow \Re$. Now f has

- An *absolute maximum* (or global maximum) value at point $c \in \mathcal{D}$ if

$$f(x) \leq f(c), \ \forall x \in \mathcal{D}$$

- An *absolute minimum* (or global minimum) value at $c \in \mathcal{D}$ if

$$f(x) \geq f(c), \ \forall x \in \mathcal{D}$$

- A *local maximum value* at c if there is an open interval $\mathcal{I}$ containing c in which $f(c) \geq f(x), \ \forall x \in \mathcal{I}$
- A *local minimum value* at c if there is an open interval $\mathcal{I}$ containing c in which $f(c) \leq f(x), \ \forall x \in \mathcal{I}$
- A *local extreme value* at c, if $f(c)$ is either a local maximum or local minimum value of f in an open interval $\mathcal{I}$ with $c \in \mathcal{I}$

First Derivative Test & Extreme Value Theorem

First derivative test for local extreme value of f, when f is differentiable at the extremum.

First Derivative Test & Extreme Value Theorem

First derivative test for local extreme value of f, when f is differentiable at the extremum.

Claim

If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.

The Extreme Value Theorem

First Derivative Test & Extreme Value Theorem

First derivative test for local extreme value of f, when f is differentiable at the extremum.

Claim

If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.

The Extreme Value Theorem

Claim

A continuous function $f(x)$ on a closed and bounded interval $[a, b]$ attains a minimum value $f(c)$ for some $c \in [a, b]$ and a maximum value $f(d)$ for some $d \in [a, b]$. That is, a continuous function on a closed, bounded interval attains a minimum and a maximum value.

We must point out that either or both of the values c and d may be attained at the end points of the interval $[a, b]$.

Taylor's Theorem and n^{th} degree polynomial approximation

The n^{th} degree polynomial approximation of a function is used to prove a generalization of the mean value theorem, called the *Taylor's theorem*.

Claim

The Taylor's theorem states that if f and its first n derivatives $f', f'', \ldots, f^{(n)}$ *are continuous on the closed interval* $[a, b]$, *and differentiable on* (a, b), *then there exists a number* $c \in (a, b)$ *such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(b-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(b-a)^{n+1}$$

Mean Value Theorem $=$ *Taylor's theorem with* $n =$

Taylor's Theorem and n^{th} degree polynomial approximation

The n^{th} degree polynomial approximation of a function is used to prove a generalization of the mean value theorem, called the *Taylor's theorem*.

Claim

The Taylor's theorem states that if f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval $[a, b]$, and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(b-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(b-a)^{n+1}$$

Mean Value Theorem = Taylor's theorem with $n = 0$

Mean Value, Taylor's Theorem and words of caution

Note that if f fails to be differentiable at even one number in the interval, then the conclusion of the mean value theorem may be false. For example, if $f(x) = x^{2/3}$, then $f'(x) = \frac{2}{3\sqrt[3]{x}}$ and

Mean Value, Taylor's Theorem and words of caution

Note that if f fails to be differentiable at even one number in the interval, then the conclusion of the mean value theorem may be false. For example, if $f(x) = x^{2/3}$, then $f'(x) = \frac{2}{3\sqrt[3]{x}}$ and the theorem does not hold in the interval $[-3, 3]$, since f is not differentiable at 0 as can be seen in Figure 1.

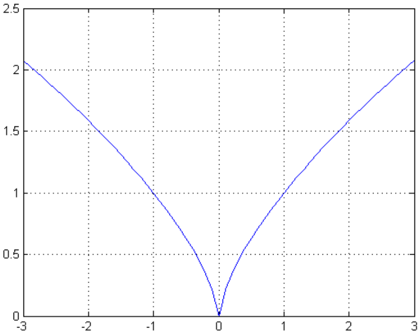


Figure 1:

Sufficient Conditions for Increasing and decreasing functions

A function f is said to be ...

- *increasing* on an interval $\mathcal{I}$ in its domain $\mathcal{D}$ if $f(t) < f(x)$ whenever $t < x$.
- *decreasing* on an interval $\mathcal{I} \in \mathcal{D}$ if $f(t) > f(x)$ whenever $t < x$.

Consequently:

Claim

Let $\mathcal{I}$ be an interval and suppose f is continuous on $\mathcal{I}$ and differentiable on $int(\mathcal{I})$. Then:

1. *if $f'(x) > 0$ for all $x \in int(\mathcal{I})$, then f is*

Sufficient Conditions for Increasing and decreasing functions

A function f is said to be ...

- *increasing* on an interval $\mathcal{I}$ in its domain $\mathcal{D}$ if $f(t) < f(x)$ whenever $t < x$.
- *decreasing* on an interval $\mathcal{I} \in \mathcal{D}$ if $f(t) > f(x)$ whenever $t < x$.

Consequently:

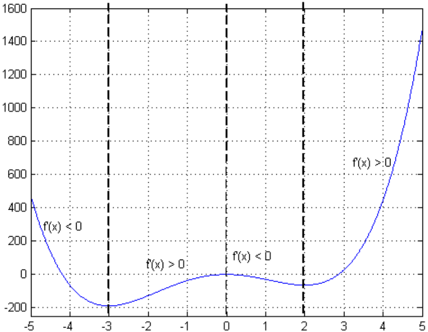
Claim

Let $\mathcal{I}$ be an interval and suppose f is continuous on $\mathcal{I}$ and differentiable on int$(\mathcal{I})$. Then:

1. *if $f'(x) > 0$ for all $x \in int(\mathcal{I})$, then f is increasing on $\mathcal{I}$;*
2. *if $f'(x) < 0$ for all $x \in int(\mathcal{I})$, then f is decreasing on $\mathcal{I}$;*
3. *if $f'(x) = 0$ for all $x \in int(\mathcal{I})$, iff, f is constant on $\mathcal{I}$.*

Illustration of Sufficient Conditions

Figure 2 illustrates the intervals in $(-\infty, \infty)$ on which the function $f(x) = 3x^4 + 4x^3 - 36x^2$ is decreasing and increasing. First we note that $f(x)$ is differentiable everywhere on $(-\infty, \infty)$ and compute $f'(x) = 12(x^3 + x^2 - 6x) = 12(x-2)(x+3)x$, which is negative in the intervals $(-\infty, -3]$ and $[0, 2]$ and positive in the intervals $[-3, 0]$ and $[2, \infty)$. We observe that f is decreasing in the intervals $(-\infty, -3]$ and $[0, 2]$ and while it is increasing in the intervals $[-3, 0]$ and $[2, \infty)$.



Necessary conditions for increasing/decreasing function

The conditions for increasing and decreasing properties of $f(x)$ stated so far are

Necessary conditions for increasing/decreasing function

The conditions for increasing and decreasing properties of $f(x)$ stated so far are not necesssary.

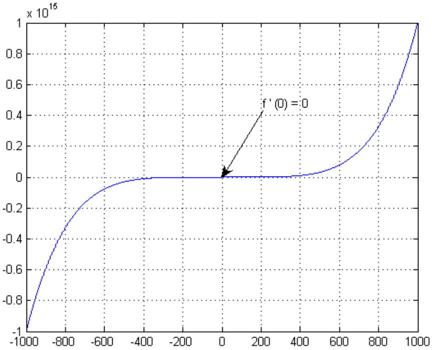


Figure 3:

Figure 3 shows that for the function $f(x) = x^5$, though $f(x)$ is increasing in $(-\infty, \infty)$, $f'(0) = 0$.

Another sufficient condition for increasing/decreasing function

Thus, a modified sufficient condition for a function f to be increasing/decreasing on an interval $\mathcal{I}$ can be stated as follows:

Another sufficient condition for increasing/decreasing function

Thus, a modified sufficient condition for a function f to be increasing/decreasing on an interval $\mathcal{I}$ can be stated as follows:

Claim

Let $\mathcal{I}$ be an interval and suppose f is continuous on $\mathcal{I}$ and differentiable on $int(\mathcal{I})$. Then:

1. *if $f'(x) \geq 0$ for all $x \in int(\mathcal{I})$, and if $f'(x) = 0$ at only finitely many $x \in \mathcal{I}$, then f is increasing on $\mathcal{I}$;*
2. *if $f'(x) \leq 0$ for all $x \in int(\mathcal{I})$, and if $f'(x) = 0$ at only finitely many $x \in \mathcal{I}$, then f is decreasing on $\mathcal{I}$.*

For example, the derivative of the function $f(x) = 6x^5 - 15x^4 + 10x^3$ vanishes at 0, and 1 and $f'(x) > 0$ elsewhere. So $f(x)$ is increasing on $(-\infty, \infty)$.

Necessary conditions for increasing/decreasing function (contd.)

We have a slightly different necessary condition..

Claim

Let $\mathcal{I}$ be an interval, and suppose f is continuous on $\mathcal{I}$ and differentiable in int$(\mathcal{I})$. Then:

1. *if f is increasing on $\mathcal{I}$, then $f'(x) \geq 0$ for all $x \in$ int$(\mathcal{I})$;*
2. *if f is decreasing on $\mathcal{I}$, then $f'(x) \leq 0$ for all $x \in$ int$(\mathcal{I})$.*

Critical Point

This concept will help us derive the general condition for local extrema.

Definition

[Critical Point]: *A point c in the domain $\mathcal{D}$ of f is called a critical point of f if either $f'(c) = 0$ or $f'(c)$ does not exist.*

The following general condition for local extrema extends the result in theorem 1 to general non-differentiable functions.

Claim

If $f(c)$ is a local extreme value, then c is a critical number of f.

The converse of above statement does not hold (see Figure 3); 0 is a critical number ($f'(0) = 0$), although $f(0)$ is not a local extreme value.

Critical Point and Local Extreme Value

Given a critical point c, the following test helps determine if $f(c)$ is a local extreme value:

Procedure

[Local Extreme Value]: *Let c be an isolated critical point of f*

1. *$f(c)$ is a local minimum if $f(x)$ is decreasing in an interval $[c - \epsilon_1, c]$ and increasing in an interval $[c, c + \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ is increasing in an interval $[c - \epsilon_1, c]$ and decreasing in an interval $[c, c + \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*

First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has

First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has the derivative $f'(x) = 15x^2(x+1)(x-1)$. The critical points are

First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has the derivative $f'(x) = 15x^2(x+1)(x-1)$. The critical points are 0, 1 and -1. Of the three, the sign of $f'(x)$ changes at 1 and -1, which are local minimum and maximum respectively. The sign does not change at 0, which is therefore not a local supremum.

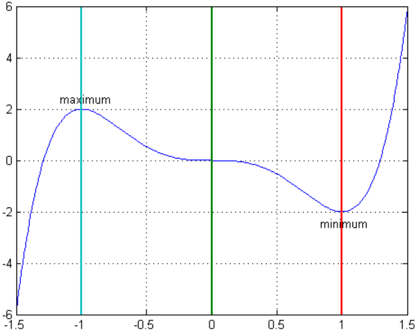


Figure 4:

First Derivative Test: Critical Point and Local Extreme Value

As another example, consider the function

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then,

First Derivative Test: Critical Point and Local Extreme Value

As another example, consider the function

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then,

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Note that $f(x)$ is discontinuous at $x = 0$, and therefore $f'(x)$ is not defined at $x = 0$. All numbers $x \geq 0$ are critical numbers. $f(0) = 0$ is a local minimum, whereas $f(x) = 1$ is a local minimum as well as a local maximum $\forall x > 0$.

Strict Convexity and Extremum

- A differentiable function f is said to be *strictly convex* (or *strictly concave up*) on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is increasing on $\mathcal{I}$.

Strict Convexity and Extremum

- A differentiable function *f* is said to be *strictly convex* (or *strictly concave up*) on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is increasing on $\mathcal{I}$.
- Recall the graphical interpretation of the first derivative $f'(x)$; $f'(x) > 0$ implies that $f(x)$ is increasing at x.
- Similarly, $f'(x)$ is increasing when

Strict Convexity and Extremum

- A differentiable function f is said to be *strictly convex* (or *strictly concave up*) on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is increasing on $\mathcal{I}$.
- Recall the graphical interpretation of the first derivative $f'(x)$; $f'(x) > 0$ implies that $f(x)$ is increasing at x.
- Similarly, $f'(x)$ is increasing when $f''(x) > 0$. This gives us a sufficient condition for the strict convexity of a function:

Claim

If at all points in an open interval $\mathcal{I}$, $f(x)$ is doubly differentiable and if $f''(x) > 0$, $\forall x \in \mathcal{I}$, then the slope of the function is always increasing with x and the graph is strictly convex. This is illustrated in Figure 5.

- On the other hand, if the function is strictly convex and doubly differentiable in $\mathcal{I}$, then $f''(x) \geq 0$, $\forall x \in \mathcal{I}$.

Strict Convexity and Extremum (Illustrated)

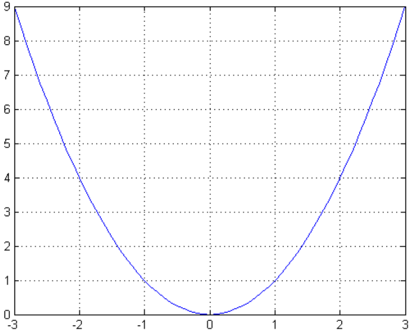


Figure 5:

Strict Convexity and Extremum: Slopeless interpretation (SI)

Claim

A function f is strictly convex on an open interval $\mathcal{I}$, iff

$$f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2) \tag{1}$$

whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ and $0 < a < 1$.

Strict Concavity

- A differentiable function f is said to be *strictly concave* on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is decreasing on $\mathcal{I}$.
- Recall from theorem 4, the graphical interpretation of the first derivative $f'(x)$; $f'(x) < 0$ implies that $f(x)$ is decreasing at x.
- Similarly, $f'(x)$ is (strictly) monotonically decreasing when

Strict Concavity

- A differentiable function f is said to be *strictly concave* on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is decreasing on $\mathcal{I}$.
- Recall from theorem 4, the graphical interpretation of the first derivative $f'(x)$; $f'(x) < 0$ implies that $f(x)$ is decreasing at x.
- Similarly, $f'(x)$ is (strictly) monotonically decreasing when $f''(x) < 0$. This gives us a sufficient condition for the concavity of a function:

Claim

If at all points in an open interval $\mathcal{I}$, $f(x)$ is doubly differentiable and if $f''(x) < 0$, $\forall x \in \mathcal{I}$, then the slope of the function is always decreasing with x and the graph is strictly concave.

Strict Concavity

On the other hand, if the function is strictly concave and doubly differentiable in $\mathcal{I}$, then $f''(x) \leq 0, \ \forall x \in \mathcal{I}$. This is illustrated in Figure 6.

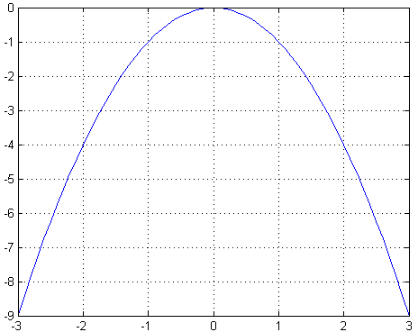


Figure 6:

Strict Concavity (slopeless interpretation)

There is also a slopeless interpretation of concavity as stated below:

Claim

A differentiable function f is strictly concave on an open interval $\mathcal{I}$, iff

$$f(ax_1 + (1-a)x_2) > af(x_1) + (1-a)f(x_2) \quad (2)$$

whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ and $0 < a < 1$.

The proof is similar to that for the slopeless interpretation of convexity.

Convex & Concave Regions and Inflection Point

Study the function $f(x) = x^3 - x + 2$.

Convex & Concave Regions and Inflection Point

Study the function $f(x) = x^3 - x + 2$. It's slope decreases as x increases to 0 ($f''(x) < 0$) and then the slope increases beyond $x = 0$ ($f''(x) > 0$). The point 0, where the $f''(x)$ changes sign is called the *inflection point*; the graph is strictly concave for $x < 0$ and strictly convex for $x > 0$. See Figure 7.

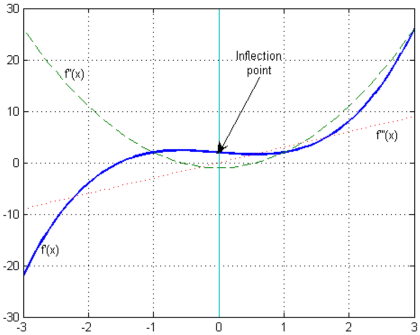


Figure 7:

Convex & Concave Regions and Inflection Point

Along similar lines, study the function $f(x) = \frac{1}{20}x^5 - \frac{7}{12}x^4 + \frac{7}{6}x^3 - \frac{15}{2}x^2$.

Convex & Concave Regions and Inflection Point

Along similar lines, study the function $f(x) = \frac{1}{20}x^5 - \frac{7}{12}x^4 + \frac{7}{6}x^3 - \frac{15}{2}x^2$.
It is strictly concave on $(-\infty, -1]$ and $[3, 5]$ and strictly convex on $[-1, 3]$ and $[5, \infty]$.
The inflection points for this function are at $x = -1$, $x = 3$ and $x = 5$.

First Derivative Test: Restated using Strict Convexity

The *first derivative test* for local extrema can be restated in terms of strict convexity and concavity of functions.

First Derivative Test: Restated using Strict Convexity

The *first derivative test* for local extrema can be restated in terms of strict convexity and concavity of functions.

Procedure

[First derivative test in terms of strict convexity]: *Let c be a critical number of f and* $f'(c) = 0$. *Then,*

1. $f(c)$ *is a local minimum if*

First Derivative Test: Restated using Strict Convexity

The *first derivative test* for local extrema can be restated in terms of strict convexity and concavity of functions.

Procedure

[First derivative test in terms of strict convexity]: *Let c be a critical number of f and* $f'(c) = 0$. *Then,*

1. $f(c)$ *is a local minimum if the graph of* $f(x)$ *is strictly convex on an open interval containing c.*
2. $f(c)$ *is a local maximum if the graph of* $f(x)$ *is strictly concave on an open interval containing c.*

Strict Convexity: Restated using Second Derivative

If the second derivative $f''(c)$ exists, then the strict convexity conditions for the critical number can be stated in terms of the sign of of $f''(c)$, making use of previous results. This is called the *second derivative test*.

Strict Convexity: Restated using Second Derivative

If the second derivative $f''(c)$ exists, then the strict convexity conditions for the critical number can be stated in terms of the sign of of $f''(c)$, making use of previous results. This is called the *second derivative test*.

Procedure

[Second derivative test]: *Let c be a critical number of f where $f'(c) = 0$ and $f''(c)$ exists.*

1. *If $f''(c) > 0$ then*

Strict Convexity: Restated using Second Derivative

If the second derivative $f''(c)$ exists, then the strict convexity conditions for the critical number can be stated in terms of the sign of of $f''(c)$, making use of previous results. This is called the *second derivative test*.

Procedure

[Second derivative test]: *Let c be a critical number of f where $f'(c) = 0$ and $f''(c)$ exists.*

1. *If $f''(c) > 0$ then $f(c)$ is a local minimum.*
2. *If $f''(c) < 0$ then $f(c)$ is a local maximum.*
3. *If $f''(c) = 0$ then $f(c)$ could be a local maximum, a local minimum, neither or both. That is, the test fails.*

Convexity, Minima and Maxima: Illustrations

Study the functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$:

Convexity, Minima and Maxima: Illustrations

Study the functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$:

- If $f(x) = x^4$, then $f'(0) = 0$ and $f''(0) = 0$ and we can see that $f(0)$ is a local minimum.
- If $f(x) = -x^4$, then $f'(0) = 0$ and $f''(0) = 0$ and we can see that $f(0)$ is a local maximum.
- If $f(x) = x^3$, then $f'(0) = 0$ and $f''(0) = 0$ and we can see that $f(0)$ is neither a local minimum nor a local maximum. $(0, 0)$ is an inflection point in this case.

Convexity, Minima and Maxima: Illustrations (contd.)

Study the functions: $f(x) = x + 2\sin x$ and $f(x) = x + \frac{1}{x}$:

- If $f(x) = x + 2\sin x$, then $f'(x) = 1 + 2\cos x$. $f'(x) = 0$ for $x = \frac{2\pi}{3}, \frac{4\pi}{3}$, which are the critical numbers. $f''\left(\frac{2\pi}{3}\right) = -2\sin\frac{2\pi}{3} = -\sqrt{3} < 0 \Rightarrow f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3}$ is a local maximum value. On the other hand, $f''\left(\frac{4\pi}{3}\right) = \sqrt{3} > 0 \Rightarrow f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum value.
- If $f(x) = x + \frac{1}{x}$, then $f'(x) = 1 - \frac{1}{x^2}$. The critical numbers are $x = \pm 1$. Note that $x = 0$ is not a critical number, even though $f'(0)$ does not exist, because 0 is not in the domain of f. $f''(x) = \frac{2}{x^3}$. $f''(-1) = -2 < 0$ and therefore $f(-1) = -2$ is a local maximum. $f''(1) = 2 > 0$ and therefore $f(1) = 2$ is a local minimum.

Global Extrema on Closed Intervals

Recall the extreme value theorem. A consequence is that:

- if either of c or d lies in (a, b), then it is a critical number of f;
- else each of c and d must lie on one of the boundaries of $[a, b]$.

This gives us a procedure for finding the maximum and minimum of a continuous function f on a closed bounded interval $\mathcal{I}$:

Procedure

[Finding extreme values on closed, bounded intervals]:

1. *Find the critical points in* $int(\mathcal{I})$.
2. *Compute the values of f at the critical points and at the endpoints of the interval.*
3. *Select the least and greatest of the computed values.*

Global Extrema on Closed Intervals (contd)

- To compute the maximum and minimum values of $f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$,

Global Extrema on Closed Intervals (contd)

- To compute the maximum and minimum values of $f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$,
  - We first compute $f'(x) = 12x^2 - 16x + 5$ which is 0 at $x = \frac{1}{2}, \frac{5}{6}$.
  - Values at the critical points are $f(\frac{1}{2}) = 1$, $f(\frac{5}{6}) = \frac{25}{27}$.
  - The values at the end points are $f(0) = 0$ and $f(1) = 1$.
  - Therefore, the minimum value is $f(0) = 0$ and the maximum value is $f(1) = f(\frac{1}{2}) = 1$.

Global Extrema on Closed Intervals (contd)

- To compute the maximum and minimum values of $f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$,
  - We first compute $f'(x) = 12x^2 - 16x + 5$ which is 0 at $x = \frac{1}{2}, \frac{5}{6}$.
  - Values at the critical points are $f(\frac{1}{2}) = 1$, $f(\frac{5}{6}) = \frac{25}{27}$.
  - The values at the end points are $f(0) = 0$ and $f(1) = 1$.
  - Therefore, the minimum value is $f(0) = 0$ and the maximum value is $f(1) = f(\frac{1}{2}) = 1$.
- In this context, it is relevant to discuss the one-sided derivatives of a function at the endpoints of the closed interval on which it is defined.

Global Extrema on Closed Intervals (contd)

Definition

[One-sided derivatives at endpoints]: *Let f be defined on a closed bounded interval $[a, b]$. The (right-sided) derivative of f at $x = a$ is defined as*

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Similarly, the (left-sided) derivative of f at $x = b$ is defined as

$$f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

Essentially, each of the one-sided derivatives defines one-sided slopes at the endpoints.

Global Extrema on Closed Intervals (contd)

Based on these definitions, the following result can be derived.

Claim

If f is continuous on $[a, b]$ and $f'(a)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at a.

- *If $f(a)$ is the maximum value of f on $[a, b]$, then $f'(a) \leq 0$ or $f'(a) = -\infty$.*
- *If $f(a)$ is the minimum value of f on $[a, b]$, then $f'(a) \geq 0$ or $f'(a) = \infty$.*

If f is continuous on $[a, b]$ and $f'(b)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at b

Global Extrema on Closed Intervals (contd)

Based on these definitions, the following result can be derived.

Claim

If f is continuous on $[a, b]$ and $f'(a)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at a.

- *If $f(a)$ is the maximum value of f on $[a, b]$, then $f'(a) \leq 0$ or $f'(a) = -\infty$.*
- *If $f(a)$ is the minimum value of f on $[a, b]$, then $f'(a) \geq 0$ or $f'(a) = \infty$.*

If f is continuous on $[a, b]$ and $f'(b)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at b

- *If $f(b)$ is the maximum value of f on $[a, b]$, then $f'(b) \geq 0$ or $f'(b) = \infty$.*
- *If $f(b)$ is the minimum value of f on $[a, b]$, then $f'(b) \leq 0$ or $f'(b) = -\infty$.*

Global Extrema on Closed Intervals (contd)

The following result gives a useful procedure for finding **extrema on closed intervals**.

Claim

If f is continuous on $[a, b]$ and $f''(x)$ exists for all $x \in (a, b)$. Then,

- *If $f''(x) \leq 0, \ \forall x \in (a, b)$, then the minimum value of f on $[a, b]$ is either $f(a)$ or $f(b)$. If, in addition, f has a critical point $c \in (a, b)$, then $f(c)$ is the maximum value of f on $[a, b]$.*
- *If $f''(x) \geq 0, \ \forall x \in (a, b)$, then the maximum value of f on $[a, b]$ is either $f(a)$ or $f(b)$. If, in addition, f has a critical point $c \in (a, b)$, then $f(c)$ is the minimum value of f on $[a, b]$.*

Global Extrema on Open Intervals

The next result is very useful for finding **extrema on open intervals.**

Claim

Let $\mathcal{I}$ be an open interval and let $f'(x)$ exist $\forall x \in \mathcal{I}$.

- *If $f'(x) \geq 0, \ \forall x \in \mathcal{I}$, and if there is a number $c \in \mathcal{I}$ where $f'(c) = 0$, then $f(c)$ is the global minimum value of f on $\mathcal{I}$.*
- *If $f'(x) \leq 0, \ \forall x \in \mathcal{I}$, and if there is a number $c \in \mathcal{I}$ where $f'(c) = 0$, then $f(c)$ is the global maximum value of f on $\mathcal{I}$.*

For example, let $f(x) = \frac{2}{3}x - \sec x$ and
$\mathcal{I} = (\frac{-\pi}{2}, \frac{\pi}{2})$.

Global Extrema on Open Intervals

The next result is very useful for finding **extrema on open intervals.**

Claim

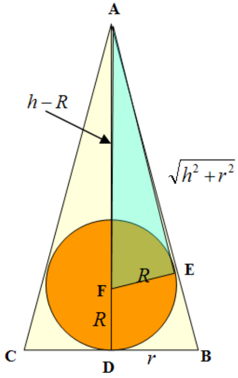
Let $\mathcal{I}$ be an open interval and let $f''(x)$ exist $\forall x \in \mathcal{I}$.

- *If $f''(x) \geq 0$, $\forall x \in \mathcal{I}$, and if there is a number $c \in \mathcal{I}$ where $f'(c) = 0$, then $f(c)$ is the global minimum value of f on $\mathcal{I}$.*
- *If $f''(x) \leq 0$, $\forall x \in \mathcal{I}$, and if there is a number $c \in \mathcal{I}$ where $f'(c) = 0$, then $f(c)$ is the global maximum value of f on $\mathcal{I}$.*

For example, let $f(x) = \frac{2}{3}x - \sec x$ and $\mathcal{I} = (\frac{-\pi}{2}, \frac{\pi}{2})$.$f'(x) = \frac{2}{3} - \sec x \tan x = \frac{2}{3} - \frac{\sin x}{\cos^2 x} = 0 \Rightarrow x = \frac{\pi}{6}$. Further, $f''(x) = -\sec x(\tan^2 x + \sec^2 x) < 0$ on $(\frac{-\pi}{2}, \frac{\pi}{2})$. Therefore, f attains the maximum value $f(\frac{\pi}{6}) = \frac{\pi}{9} - \frac{2}{\sqrt{3}}$ on $\mathcal{I}$.

Global Extrema on Open Intervals (contd)

As another example, let us find the dimensions of the cone with minimum volume that can contain a sphere with radius R. Let h be the height of the cone and r the radius of its base. The objective to be minimized is the volume $f(r,h) = \frac{1}{3}\pi r^2 h$. The constraint betwen r and h is shown in Figure 8. The traingle AEF is similar to traingle ADB and therefore, $\frac{h-R}{R} = \frac{\sqrt{h^2+r^2}}{r}$.



Global Extrema on Open Intervals (contd)

Our first step is to reduce the volume formula to involve only one of r^2[1] or h.
The algebra involved will be the simplest if we solved for h.
The constraint gives us $r^2 = \frac{R^2 h}{h-2R}$. Substituting this expression for r^2 into the volume formula, we get $g(h) = \frac{\pi R^2}{3} \frac{h^2}{(h-2R)}$ with the domain given by $\mathcal{D} = \left\{ h | 2R < h < \infty \right\}$.
Note that $\mathcal{D}$ is an open interval.
$g' = \frac{\pi R^2}{3} \frac{2h(h-2R)-h^2}{(h-2R)^2} = \frac{\pi R^2}{3} \frac{h(h-4R)}{(h-2R)^2}$ which is 0 in its domain $\mathcal{D}$ if and only if $h = 4R$.
$g'' = \frac{\pi R^2}{3} \frac{2(h-2R)^3-2h(h-4R)(h-2R)^2}{(h-2R)^4} = \frac{\pi R^2}{3} \frac{2(h^2-4Rh+4R^2-h^2+4Rh)}{(h-2R)^3} = \frac{\pi R^2}{3} \frac{8R^2}{(h-2R)^3}$, which is greater than 0 in $\mathcal{D}$.
Therefore, g (and consequently f) has a unique minimum at $h = 4R$ and correspondingly, $r^2 = \frac{R^2 h}{h-2R} = 2R^2$.

[1] Since r appears in the volume formula only in terms of r^2.

From $\Re$ to $\Re^n$.

Local Extrema for $f(x_1, x_2.., x_n)$

Definition

[Local minimum]: *A function* $f : \mathcal{D} \rightarrow \Re$ *of n variables has a local minimum at* $\mathbf{x}^0$ *if* $\exists \mathcal{N}(\mathbf{x}^0)$ *such that* $\forall\ \mathbf{x} \in \mathcal{N}(\mathbf{x}^0)$, $f(\mathbf{x}^0) \leq f(\mathbf{x})$. *In other words,* $f(\mathbf{x}^0) \leq f(\mathbf{x})$ *whenever* $\mathbf{x}$ *lies in some neighborhood around* $\mathbf{x}^0$. *An example neighborhood is the circular disc when* $\mathcal{D} = \Re^n$.

Definition

[Local maximum]: .... $f(\mathbf{x}^0) \geq f(\mathbf{x})$.

General Reference: Stories About Maxima and Minima (Mathematical World) by Vladimir M. Tikhomirov

Local Extrema

These definitions are exactly analogous to the definitions for a function of single variable. Figure 9 shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.

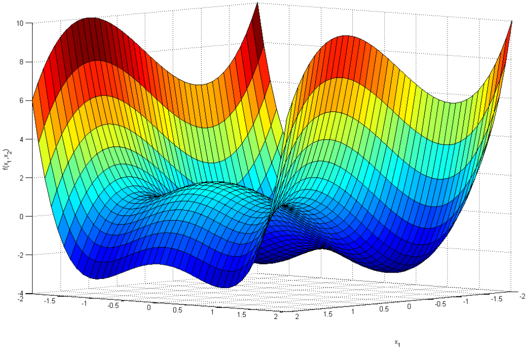


Figure 9:

Convexity and Extremum: Slopeless interpretation (SI)

Definition

A function f is convex on $\mathcal{D}$, *iff*

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \tag{3}$$

and is strictly convex on $\mathcal{D}$, *iff*

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \tag{4}$$

whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < \alpha < 1$.

Note: This implicitly assumes that whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$,

Convexity and Extremum: Slopeless interpretation (SI)

Definition

A function f is convex on $\mathcal{D}$, *iff*

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \tag{3}$$

and is strictly convex on $\mathcal{D}$, *iff*

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \tag{4}$$

whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < \alpha < 1$.

Note: This implicitly assumes that whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, $\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \mathcal{D}$

Local Extrema

Figure 10 shows the plot of $f(x_1, x_2) = 3x_1^2 + 3x_2^2 - 9$. As can be seen in the plot, the function is cup shaped and appears to be convex everywhere in $\Re^2$.

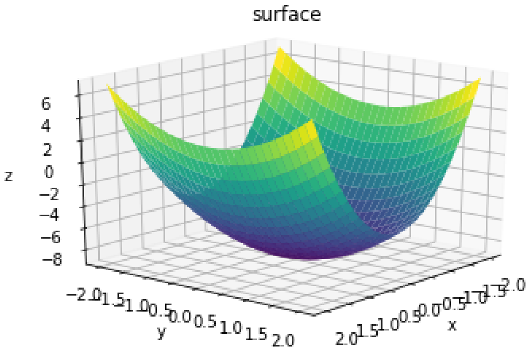


Figure 10:

From $f(x) : \Re \rightarrow \Re$ to $f(x_1, x_2 \ldots x_n) : \mathcal{D} \rightarrow \Re$

Need to also extend

- Extreme Value Theorem
- Rolle's theorem, Mean Value Theorem, Taylor Expansion
- Necessary and Sufficient first and second order conditions for local/extrema
- First and second order conditions for Convexity

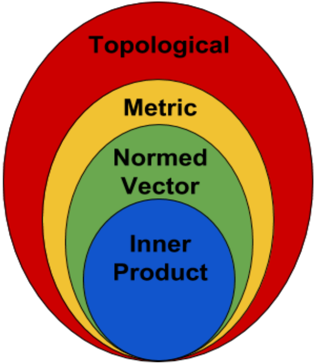
Need following notions/definitions in $\mathcal{D}$

- Neighborhood and open sets/balls ($\Leftarrow$ Local extremum)
- Bounded, Closed Sets ($\Leftarrow$ Extreme value theorem)
- Convex Sets ($\Leftarrow$ Convex functions of n variables)
- Directional Derivatives and Gradients ($\Leftarrow$ Taylor Expansion, all first order conditions)

Spaces The Mathematical Structures)

Contents: The Mathematical Structures called Spaces

- Topological Spaces: Notion of neighbourhood of points.
- Metric Spaces: Notion of positive distance between two points.
- Normed Vector Spaces: Notion of positive length of each point.
- Inner Product Spaces: Notion of projection of one point on another, both positive and negative.



Topological Spaces

Set of points X along with the set of neighbors ($N(\mathbf{x})$) of each point ($\mathbf{x} \in X$), with certain axioms required to be satisfied by the points and their neighbors.

- Definition 1: A topological space is an ordered pair $(X,\mathcal{N})$, where:
  - X is a set
  - $\mathcal{N}$ is a collection of subsets of X, satisfying the following axioms:
    - ★ The empty set and X itself belong to $\mathcal{N}$.
    - ★ Any (finite or infinite) union of members of $\mathcal{N}$ still belongs to $\mathcal{N}$.
    - ★ The intersection of any finite number of members of $\mathcal{N}$ still belongs to $\mathcal{N}$.

- As per above example, which out of following are toplogies with $X = \{1,2,3\}$ and $\mathcal{N} =$
  - $\{\{\},\{1,2,3\}\}$
  - $\{\{\},\{1\},\{1,2,3\}\}$
  - $\{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$
  - $\{\{\},\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$
  - $\{\{\},\{1\},\{2\},\{1,2,3\}\}$
  - $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$

Metric Spaces

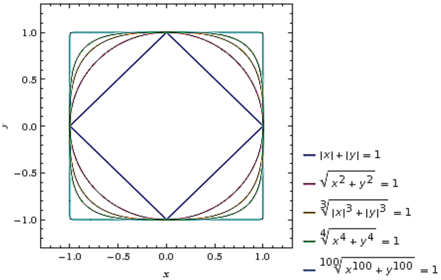
Set of points (X) along with a notion of distance $d(\mathbf{x}_1, \mathbf{x}_2)$ between any two points$(\mathbf{x}_1, \mathbf{x}_2 \in X)$ such that:

1. $d(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ (non-negativity).
2. $d(\mathbf{x}_1, \mathbf{x}_2) = 0$ iff $\mathbf{x}_1 = \mathbf{x}_2$ (identity).
3. $d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{x}_2, \mathbf{x}_1)$ (symmetry).
4. $d(\mathbf{x}_1, \mathbf{x}_2) + d(\mathbf{x}_2, \mathbf{x}_3) \geq d(\mathbf{x}_1, \mathbf{x}_3)$ (triangle inequality).

Metric Spaces

Examples:

- 1-metric d_1: The plane with the taxi cab metric
  - $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$
- 2-metric d_2: The plane $\Re^2$ with the 'usual distance' (measured using Pythagoras's theorem):
  - $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- Infinity metric d_∞: The plane with the maximum metric
  - $d((x_1, y_1), (x_2, y_2)) = max(|x_1 - x_2|, |y_1 - y_2|)$



Normed Vector Spaces

- **Vector Space:** A space consisting of vectors, together with the
  1. **associative and commutative operation of addition of vectors**,
  2. **associative and distributive operation of multiplication of vectors by scalars.**
- **Norm:** A function that assigns a strictly positive length or size to each vector in a vector space — save for the zero vector, which is assigned a length of zero.
- **Normed Vector Space:** A vector space on which a norm is defined.

Normed Vector Spaces

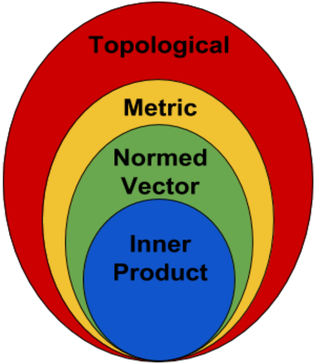
A vector space on which a norm is defined.

- In any real vector space $\Re^n$, the length of a vector has the following properties:
  1. The zero vector, 0, has zero length; every other vector has a positive length.
     - ★ $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. Multiplying a vector by a positive number changes its length without changing its direction. Moreover,
     - ★ $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
  3. The triangle inequality holds. That is, taking norms as distances, the distance from point A through B to C is never shorter than going directly from A to C, or the shortest distance between any two points is a straight line.
     - ★ $||\mathbf{x}_1 + \mathbf{x}_2|| \leq ||\mathbf{x}_1|| + ||\mathbf{x}_2||$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

The generalization of these three properties to more abstract vector spaces leads to the notion of norm. For example: A matrix norm.
Additionally, in the case of square matrices (thus, m $=$ n), some (but not all) matrix norms satisfy the following condition, which is related to the fact that matrices are more than just vectors: $||\mathbf{AB}|| \leq ||\mathbf{A}||\ ||\mathbf{B}||$ **for all matrices A and B in** K^{nxn}**.**

Contrasting the Spaces discussed so far

- Topological Spaces: Notion of neighbourhood of points.
- Metric Spaces: Notion of positive distance between two points.
- Normed Vector Spaces: Notion of positive length of each point.



Topological Spaces

Set of points X along with the set of open sets ($\mathcal{N}$) with certain axioms required to be satisfied by sets in $\mathcal{N}$:

- Definition 1: A topological space is an ordered pair (X,$\mathcal{N}$), where:
  - X is a set
  - $\mathcal{N}$ is a collection of subsets of X, satisfying the following axioms:
    - ★ The empty set and X itself belong to $\mathcal{N}$.
    - ★ Any (finite or infinite) union of members of $\mathcal{N}$ still belongs to $\mathcal{N}$.
    - ★ The intersection of any finite number of members of $\mathcal{N}$ still belongs to $\mathcal{N}$.
- We already saw examples that are (and are not) toplogies for $X = \{1,2,3\}$ and $\mathcal{N} =$
  - $\{\{\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{1\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{1\},\{2\},\{1,2,3\}\}$ **No as {1}** $\cup$ **{2}** $\notin \mathcal{N}$
  - $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$ **No as {1,2}** $\cap$ **{2,3}** $\notin \mathcal{N}$

Topological Spaces and Open Sets

The neighbourhoods can be recovered by defining $N(\mathbf{x})$ to be a neighbourhood of $\mathbf{x}$ if $\mathcal{N}$ includes a set O such that $\mathbf{x} \in O$. The sets $O \in \mathcal{N}$ are basically the open sets. For example

- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\},\{1,2,3\}$, each of $\{\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1,2,3\}\}$
  $N(2) \in \{\{1,2,3\}\}$
  $N(3) \in \{\{1,2,3\}\}$
- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\},\{1\},\{1,2,3\}\}$, each of $\{\}$,$\{1\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1\},\{1,2,3\}\}$
  $N(2) \in \{\{1,2,3\}\}$
  $N(3) \in \{\{1,2,3\}\}$
- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$, each of $\{\},\{1\},\{2\},\{1,2\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1\},\{1,2\},\{1,2,3\}\}$
  $N(2) \in \{\{2\},\{1,2\},\{1,2,3\}\}$
  $N(3) \in \{\{1,2,3\}\}$

Topological Spaces and Open Sets

- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\},\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$ each of $\{\},\{2\},\{1,2\},\{2,3\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1,2\},\{1,2,3\}\}$
  $N(2) \in \{\{1,2\},\{2,3\},\{1,2,3\}\}$
  $N(3) \in \{\{2,3\},\{1,2,3\}\}$

(Alternative) Definition 2: A topological space is an ordered pair $(X,N(.))$, where X is a set and $N(.)$ is a neighborhood function such that for each $\mathbf{x} \in X$, if $N(\mathbf{x})$ is a

- neighbourhood of $\mathbf{x}$ then $\mathbf{x} \in N(\mathbf{x})$.
- subset of X and includes a neighbourhood of $\mathbf{x}$, then $N(bfx)$ is a neighbourhood of $\mathbf{x}$.
- neighbourhood of $\mathbf{x}$, then for any other neighborhood $N'(\mathbf{x})$, $N(\mathbf{x}) \cap N'(\mathbf{x})$ is also a neighbourhood of $\mathbf{x}$.
- neighbourhood of $\mathbf{x}$, then it includes a neighbourhood $N'(\mathbf{x})$ such that $N(\mathbf{x})$ is a neighbourhood of each point of $N'(\mathbf{x})$.

What topological spaces (and their special cases) give us

- Definition 1: A topological space is an ordered pair $(X,\mathcal{N})$, where.....
- Definition 2: A topological space is an ordered pair $(X,N(.))$, where....
- Definition 1 allows for understanding open sets as elements of $\mathcal{N}$.
  - We can define an open ball $B(\mathbf{x})$ to be any element of $N(\mathbf{x})$.
  - If additionally, we have metric $d(.,.)$ on the space, we can define an open ball $B(\mathbf{x}, r)$ of radius r as $\{\mathbf{y}|\mathbf{d}(\mathbf{x},\mathbf{y}) < \mathbf{r}\}$
  - A norm ball $B(\mathbf{x}, r) = \{\mathbf{y}|\|\mathbf{x}-\mathbf{y}\| < \mathbf{r}\}$ also should have homogenity! That is, $\|\alpha\mathbf{x} - \alpha\mathbf{y}\| = \alpha\|\mathbf{x}-\mathbf{y}\|$
- Definition 2 allows for continuity of function f definied from a topology $X, N(.)$ to another topology $Y, M(.)$. Function f is continuous if for every $\mathbf{x} \in X$ and every neighbourhood $M(f(\mathbf{x}))$ of $f(\mathbf{x})$ there is a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $f(N(\mathbf{x})) \subseteq M(\mathbf{x})$.

HW1: A Topological space that does not have metric

Consider $X = \{0, 1\}$ and $\mathcal{N} = \{\emptyset, \{0\}, \{0, 1\}\}$,
Consider some metric $d(., .)$ which is 0 if both its arguments are the same and 1 otherwise. If d would be such a metric, a neighborhood (ball) of radius 0.5 around 1, that is $B(1, 0.5)$ would equal $\{1\}$, which should have been open. However, $\{1\} \notin \mathcal{N}$. Contradiction!

HW2: A metric space that does not have norm

Consider (again) the **discrete** metric $d(.,.)$ over a vector space V. We define $d(.,.)$ to be 0 if both its arguments are the same and 1 otherwise. While one can verify that this metric satisifies the triangle inequality, what one requires from an equivalent norm $\|.\|_n$ is that for any $\mathbf{x},\mathbf{y} \in V$, with $\mathbf{x} \neq \mathbf{y}$, for any scalar $\alpha \neq 0$, we must have $\|\alpha\mathbf{x} - \alpha\mathbf{y}\|_n = \alpha\|\mathbf{x} - \mathbf{y}\|_n$. This measure using the norm can clearly not correspond to the **discrete** distance metric.

Inner Product Space

It is a vector space over a field of scalars along with an inner product.

- **Field of scalars:** e.g. IR algebraic structure with:-
  1. Addition: must be multiplicative and associative.
  2. Subtraction.
  3. Multiplication: must be commutative, associative and distributive.
  4. Division: multiplicative inverse must exist.
- **Inner Product:**
  1. (Conjugate) Symmetry: $<\mathbf{x}_1, \mathbf{x}_2> = \overline{<\mathbf{x}_2, \mathbf{x}_1>}$.
  2. Linearity in the first argument.
     - $< a\mathbf{x}_1, \mathbf{x}_2 >= a < \mathbf{x}_1, \mathbf{x}_2 >$
     - $< \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_3 >=< \mathbf{x}_1, \mathbf{x}_3 > + < \mathbf{x}_3, \mathbf{x}_3 >$
  3. Positive definiteness: $< \mathbf{x}, \mathbf{x} >\geq 0$, with equality iff $\mathbf{x} = 0$.

Proof: Normed Vector Space is a Metric Space

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
   1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
   2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
   3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Metric Space: Set of points (X) along with a notion of distance $d(\mathbf{x}_1, \mathbf{x}_2)$ between any two points $(\mathbf{x}_1, \mathbf{x}_2 \in X)$ such that:
   1. $d(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ (non-negativity).
   2. $d(\mathbf{x}_1, \mathbf{x}_2) = 0$ iff $\mathbf{x}_1 = \mathbf{x}_2$ (identity).
   3. $d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{x}_2, \mathbf{x}_1)$ (symmetry).
   4. $d(\mathbf{x}_1, \mathbf{x}_2) + d(\mathbf{x}_2, \mathbf{x}_3) \geq d(\mathbf{x}_1, \mathbf{x}_3)$ (triangle inequality).
3. Proof:

Proof: Normed Vector Space is a Metric Space

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
    1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
    2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
    3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Metric Space: Set of points (X) along with a notion of distance $d(\mathbf{x}_1, \mathbf{x}_2)$ between any two points $(\mathbf{x}_1, \mathbf{x}_2 \in X)$ such that:
    1. $d(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ (non-negativity).
    2. $d(\mathbf{x}_1, \mathbf{x}_2) = 0$ iff $\mathbf{x}_1 = \mathbf{x}_2$ (identity).
    3. $d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{x}_2, \mathbf{x}_1)$ (symmetry).
    4. $d(\mathbf{x}_1, \mathbf{x}_2) + d(\mathbf{x}_2, \mathbf{x}_3) \geq d(\mathbf{x}_1, \mathbf{x}_3)$ (triangle inequality).
3. Proof:
    1. In vector space, a vector $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ can be defined by subtraction. Define $d(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_2\|$, so 1.1 $\Rightarrow \|\mathbf{x}_1 - \mathbf{x}_2\| \geq 0$; $\|\mathbf{x}_1 - \mathbf{x}_2\| = 0$ iff $\mathbf{x}_1 - \mathbf{x}_2 = 0$, hence 2.1 and 2.2 are proved.
    2. 1.2 $\Rightarrow \|-1(\mathbf{x}_1 - \mathbf{x}_2)\| = |-1|\|\mathbf{x}_1 - \mathbf{x}_2\|$. So, $\|\mathbf{x}_2 - \mathbf{x}_1\| = \|\mathbf{x}_1 - \mathbf{x}_2\|$, so 2.3 is proved.
    3. Take $\mathbf{x}_1 = \mathbf{z}_1 - z_0$ and $\mathbf{x}_2 = \mathbf{z}_0 - \mathbf{z}_2$, put in 1.3 to get $\|\mathbf{z}_1 - \mathbf{z}_0\| + \|\mathbf{z}_0 - \mathbf{z}_2\| \geq \|\mathbf{z}_1 - \mathbf{z}_2\|$ so 2.4 is prooved.

The Mathematical Structures & Spaces: Some Proofs

Some Proofs For Mathematical Structures & Spaces

- Under what conditions on P, is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?
- Prove that inner product space is a normed vector space.
- What is an example of normed vector space that is not an inner product space?
- Prove that $| < u, v > | \leq \|u\|_P \|v\|_P$ for any norm P.

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Assume $\mathbf{x} \in \Re^n$ and $P \in \Re^{n \times n}$.

1. P is symmetric positive definite iff:
   1. Symmetric: $P^T = P$
   2. Positive Definite: $\forall \mathbf{x} \neq 0$, $\mathbf{x}^T P \mathbf{x} \geq 0$

Proof:

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Assume $\mathbf{x} \in \Re^n$ and $P \in \Re^{n \times n}$.

1. P is symmetric positive definite iff:
    1. Symmetric: $P^T = P$
    2. Positive Definite: $\forall \mathbf{x} \neq 0$, $\mathbf{x}^T P \mathbf{x} \geq 0$

Proof:

- If P is symmetric positive definite (SPD), then P can be written as:
    - $P = LDL^T$, where ...
        - ★ L is lower triangular matrix with a 1 in each diagonal entry.
        - ★ D is diagonal matrix with positive values.
- So, we can write $P = RR^T$ where $R = L\sqrt{D}$.
- Thus we have $\mathbf{x}^T P \mathbf{x} = \mathbf{x}^T R R^T \mathbf{x} = (R^T \mathbf{x})^T (R^T \mathbf{x}) = \mathbf{y}^T \mathbf{y}$
    - where $\mathbf{y} = (R^T \mathbf{x})$ and thus $\mathbf{y} \in \Re^n$.
- So, $\mathbf{x}^T P \mathbf{x} \geq 0$.

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Recall:

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
    1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
    2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
    3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Proof:

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Recall:

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
   1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
   2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
   3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Proof:

1. By definition of PST: $\|\mathbf{x}^T P\mathbf{x}\| \geq 0$, and $\|\mathbf{x}^T P\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
2. For any scalar α: $\|\alpha\mathbf{x}\|_P = \sqrt{(\alpha\mathbf{x})^T P(\alpha\mathbf{x})} = \sqrt{(\alpha^2)(\mathbf{x}^T P\mathbf{x})} = \alpha\sqrt{\mathbf{x}^T P\mathbf{x}} = |\alpha|\|\mathbf{x}\|_P$.
3. $\|\mathbf{x}_1 + \mathbf{x}_2\|_P \leq \|\mathbf{x}_1\|_P + \|\mathbf{x}_2\|_P$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$. **Next Slide.**

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Proof for $\|\mathbf{x}_1 + \mathbf{x}_2\|_P \leq \|\mathbf{x}_1\|_P \|\mathbf{x}_2\|_P$

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Proof for $\|\mathbf{x}_1 + \mathbf{x}_2\|_P \leq \|\mathbf{x}_1\|_P \|\mathbf{x}_2\|_P$

For any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$:

1. $\|\mathbf{x}_1 + \mathbf{x}_2\|_P^2 =$
   - $(\mathbf{x}_1 + \mathbf{x}_2)^T P (\mathbf{x}_1 + \mathbf{x}_2)$
   - $\mathbf{x}_1^T P \mathbf{x}_1 + \mathbf{x}_2^T P \mathbf{x}_2 + \mathbf{x}_1^T P \mathbf{x}_2 + \mathbf{x}_2^T P \mathbf{x}_1$
   - $u^T u + v^T v + u^T v + v^T u$ (Using $P = RR^T$, $u = R^T \mathbf{x}_1$ and $v = R^T \mathbf{x}_2$)
   - $u^T u + v^T v + 2u^T v$, since $u^T v = v^T u$
2. $(\|\mathbf{x}_1\|_P + \|\mathbf{x}_2\|_P)^2 =$
   - $\|\mathbf{x}_1\|_P^2 + \|\mathbf{x}_2\|_P^2 + 2\|\mathbf{x}_1\|_P \|\mathbf{x}_2\|_P$
   - $\mathbf{x}_1^T P \mathbf{x}_1 + \mathbf{x}_2^T P \mathbf{x}_2 + 2\sqrt{(\mathbf{x}_1^T P \mathbf{x}_1)(\mathbf{x}_2^T P \mathbf{x}_2)}$
   - $u^T u + v^T v + 2\sqrt{(u^T u)(v^T v)}$
3. By Cauchy Schwarz Inequality: $u^T v \leq \sqrt{(u^T u)(v^T v)}$ $(Cos(\theta) \leq 1)$

Recall: Inner Product Space

It is a vector space over a field of scalars along with an inner product.

- **Field of scalars:** e.g. IR algebraic structure with:-
  1. Addition: must be multiplicative and associative.
  2. Subtraction.
  3. Multiplication: must be commutative, associative and distributive.
  4. Division: multiplicative inverse must exist.
- **Inner Product:**
  1. (Conjugate) Symmetry: $<\mathbf{x}_1, \mathbf{x}_2> = \overline{< \mathbf{x}_2, \mathbf{x}_1 >}$.
  2. Linearity in the first argument.
     - ⋆ $< a\mathbf{x}_1, \mathbf{x}_2 >= a < \mathbf{x}_1, \mathbf{x}_2 >$
     - ⋆ $< \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_3 >=< \mathbf{x}_1, \mathbf{x}_3 > + < \mathbf{x}_3, \mathbf{x}_3 >$
  3. Positive definiteness: $< \mathbf{x}, \mathbf{x} >\geq 0$, with equality iff $\mathbf{x} = 0$.

Prove that inner product space is a normed vector space.

Q) Why field of scalers?
A) By conjugate symmetry, we have $<\mathbf{x},\mathbf{x}>=\overline{<\mathbf{x},\mathbf{x}>}$. So $<\mathbf{x},\mathbf{x}>$ must be real.
So, we can define $\|\mathbf{x}\| = \sqrt{<\mathbf{x},\mathbf{x}>}$.
We need to prove that $\|\mathbf{x}\|$ is a valid norm:-

1. By positive definiteness: $<\mathbf{x},\mathbf{x}>\geq 0$, with equality iff $\mathbf{x}=0$. So $\|\mathbf{x}\|\geq 0$ ($=$ iff $\mathbf{x}=0$).
2. For any complex t, $\|t\mathbf{x}\| = \sqrt{<t\mathbf{x},t\mathbf{x}>} = \sqrt{t*\bar{t}<\mathbf{x},\mathbf{x}>} = |t|\sqrt{<\mathbf{x},\mathbf{x}>}$ (as $|t| = \sqrt{t*\bar{t}}$) So $\|t\mathbf{x}\| == |t|\|\mathbf{x}\|$
3. $\|\mathbf{x}_1+\mathbf{x}_2\| = \sqrt{<\mathbf{x}_1+\mathbf{x}_2,\mathbf{x}_1+\mathbf{x}_2>} =$ $\sqrt{<\mathbf{x}_1,\mathbf{x}_1>+<\mathbf{x}_2,\mathbf{x}_2>+<\mathbf{x}_1,\mathbf{x}_2>+<\mathbf{x}_2,\mathbf{x}_1>}$ $\leq \sqrt{<\mathbf{x}_1,\mathbf{x}_1>+<\mathbf{x}_2,\mathbf{x}_2>+2\sqrt{<\mathbf{x}_1,\mathbf{x}_1><\mathbf{x}_2,\mathbf{x}_2>}}$ (by Cauchy Schwartz inequality)

Example of normed vector space that is not an inner product space.

$$\|\mathbf{x}\|_p = \left[\sum_{i=1}^{\infty} |\mathbf{x}_i|^p \right]^{\frac{1}{p}}$$

Prove that $| < \mathbf{u}, \mathbf{v} > | \leq \|\mathbf{u}\|_P \|v\|_P$ for any norm P

Proof:

- If $\mathbf{u} = 0$ or $\mathbf{v} = 0$, then L.H.S. $=$ R.H.S $= 0.$ Hence the equality holds.
- Assume $\mathbf{u}, \mathbf{v} \neq 0.$ Let $\mathbf{z} = \mathbf{u} - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}.$
- By linearity of inner product in first argument, we have: $< \mathbf{z}, \mathbf{v} >=< u - \frac{<u,v>}{<v,v>}\mathbf{v}, \mathbf{v} >=< \mathbf{u}, \mathbf{v} > - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>} < \mathbf{v}, \mathbf{v} >= 0$
- Therefore, $< \mathbf{u}, \mathbf{u} >=< \mathbf{z} + \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}, \mathbf{z} + \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v} >=< \mathbf{z}, \mathbf{z} > +(\frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>})^2 < \mathbf{v}, \mathbf{v} > +0$
- So $< \mathbf{u}, \mathbf{u} >\geq \frac{|<\mathbf{u},\mathbf{v}>|^2}{<\mathbf{v},\mathbf{v}>}$

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Motivation:

- Consider the following inner product on $\Re^2$: For any $\mathbf{x}, \mathbf{y} \in \Re^2$, let $< \mathbf{x}, \mathbf{y} >= 2x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. It can be easily verified that this in an inner product (by checking for linearity, symmetry and positive definiteness by expressing it as a sum of squares).
- This inner product is certainly different from the conventional (Eucledian) dot product $< \mathbf{x}, \mathbf{y} >_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $< \mathbf{x}, \mathbf{y} >$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Motivation:

- Consider the following inner product on $\Re^2$: For any $\mathbf{x}, \mathbf{y} \in \Re^2$, let $< \mathbf{x}, \mathbf{y} >= 2x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. It can be easily verified that this in an inner product (by checking for linearity, symmetry and positive definiteness by expressing it as a sum of squares).
- This inner product is certainly different from the conventional (Eucledian) dot product $< \mathbf{x}, \mathbf{y} >_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $< \mathbf{x}, \mathbf{y} >$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

In $\Re^n$, it can be proved that for any inner product vector space $(\mathcal{V}, < .,. >)$, the inner product $< .,. >$ (including the Eucledian one) can be represented as

$$< \mathbf{u}, \mathbf{v} >= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j < \mathbf{e}_i, \mathbf{e}_j >= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}^T E \mathbf{b} =< \mathbf{a}^T, \mathbf{b} >_E$$

Recap: Basis and Dimensions from Linear Algebra wrt $< .,. >_E$ (Eucledian Inner Product) (For your homework)

Instructor: Prof. Ganesh Ramakrishnan

Recap: Basis and Dimensions from Linear Algebra (For your homework)

Recap: Basis in Linear Algebra

Basis for a space: *The basis for a space is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with two properties, viz., (1) The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are independent and (2) These vectors span the space.*

Set of vectors that is necessary and sufficient for spanning the space.
Eg: A (standard) basis for the four dimensional space $\Re^4$ is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (5)$$

It is easy to verify that the above vectors are independent; if a combination of the vectors using the scalars in $[c_1, c_2, c_3, c_4]$ should yield the zero vector, we must have $c_1 = c_2 = c_3 = c_4 = 0$. Another way of proving this is by making the four vectors the columns of a matrix. The resultant matrix will be an identity matrix. The null space of an identity matrix is the zero vector.

Recap: Basis in Linear Algebra (contd.)

This is not the only basis of $\Re^4$. Consider the following three vectors

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \tag{6}$$

These vectors are certainly independent. But they do not span $\Re^4$.
This can be proved by showing that the following vector in $\Re^4$ cannot be expressed as a linear combination of these vectors.

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Recap: Basis in Linear Algebra (contd.)

- In fact, if the last vector on the previous slide is added to the set of three vectors in (6), together, they define another basis for $\Re^4$.
- This could be proved by introducing them as columns of a matrix A, subject A to row reduction and check if there are any free variables (or equivalently, whether all columns are pivot columns). If there are no free variables, we can conclude that the vectors form a basis for $\Re^4$.
- This is also equivalent to the statement that *if the matrix A is invertible, its columns form a basis for its column space*.

Recap: Basis in Linear Algebra (contd.)

- We can generalize our observations to $\Re^n$: *if an $n \times n$ matrix A is invertible, its coulumns for a basis for $\Re^n$.*
- While there can be many bases for a space, a commonality between all the bases is that they have exactly the same number of vectors.
- This unique size of the basis is called the dimension of the space.

Dimension: *The number of vectors in any basis of a vector space is called the dimension of the space.*

Recap: Basis in Linear Algebra (contd.)

Do the vectors in (6), form a basis for any space at all?
The vectors are independent and therefore span the space of all linear combinations of the three vectors.
The space spanned by these vectors is a hyperplane in $\Re^4$.
Let A be any matrix. By definition, the columns of A span the column space $C(A)$ of A. If there exists a $\mathbf{c} \neq \mathbf{0}$ such that, $A\mathbf{c} = 0$, then the columns of A are not linearly independent.
For example, the columns of the matrix A given below are not linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix} \tag{8}$$

A choice of $\mathbf{c} = [-1\ 0\ 0\ 1]^T$ gives $A\mathbf{c} = 0$. Thus, the columns of A do not form a basis for its columns space.

Recap: Basis in Linear Algebra (contd.)

What is a basis for $C(A)$? A most natural choice is the first two columns of A; the thid column is the sum of the first and second columns, while the fourth column is the same as the first column. Also, column elimination[2] on A yields pivots on the first two columns. Thus, a basis for $C(A)$ is

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \tag{9}$$

Another basis for $C(A)$ consists of the first and third columns. We note that the dimension of $C(A)$ is 2. We also note that the rank of A is the number of its pivots columns, which is exactly the dimension of $C(A)$.

[2] Column elimination operations are very similar to row elimination operations.

Recap: Basis in Linear Algebra (contd.)

All of this gives us a nice result.

Theorem

The rank of a matrix is the same as the dimension of its column space. That is, $rank(A) = dimension\left(C(A)\right)$.

- What about the dimension of the null space? We already saw that $\mathbf{c} = [-1\ 0\ 0\ 1]^T$ is in the null space.
- Another element of the null space is $\mathbf{c}' = [1\ 1\ \ -1\ 0]^T$. These vectors in the null space specify combinations of the columns that yield zeroes. The two vectors $\mathbf{c}$ and $\mathbf{c}'$ are obviously independent. Do these two vectors span the entire null space?
- The dimension of the null space is the same as the number of free variables, which happens to be $4 - 2 = 2$ in this example. Thus the two vectors c and c' must indeed span the null space. In fact, it can be proved that the dimension of the null space of an $m \times n$ matrix A is $n - rank(A)$.

Recap: Row Space and Column Space in Linear Algebra (contd.)

- The space spanned by the rows of a matrix is called the *row space*. We can also define the row space of a matrix A as the column space of its transpose A^T. Thus the row space of A can be specified as $C(A^T)$.
- The null space of A, $N(A)$ is often called the *right null space* of A, while the null space of A^T, $N(A^T)$ is often referred to as its *left null space*.
- How do we visualize these four spaces? $N(A)$ and $C(A^T)$ of an $m \times n$ matrix A are in $\Re^n$, while $C(A)$ and $N(A^T)$ are in $\Re^m$.
- How can we construct bases for each of the four subspaces? We note that dimensions of $C(A)$ and the rank of $C(A^T)$ should be the same, since row rank of a matrix is its column rank. The bases of $C(A)$ can be obtained as the set of the pivot columns.

Recap: The Four Subspaces and their Bases (contd.)

- Let r be the rank of A. Recall that the null space is constructed by linear combinations of the special solutions of the null space (**??**) and there is one special solution for each assignment of the free variables. In fact, the number of special solutions exactly equals the number of free variables, which is $n - r$. Thus, the dimension of $N(A)$ will be $n - r$.
- Similarly, the dimension of $N(A^T)$ will be $m - r$.

Let us illustrate all this on the sample matrix in (8).

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix} \overset{E_{2,1},E_{3,1}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix} \overset{E_{3,2}}{\Longrightarrow} (R =) \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Recap: The Four Subspaces and their Bases (contd.)

- The reduced matrix R has the same row space as A, by virtue of the nature of row reduction. In fact, the rows of A can be retrieved from the rows of R by reversing the linear operations involved in row elimination. The first two rows give a basis for the row space of A.
- The dimension of $C(A^T)$ is 2, which is also the rank of A.
- To find the left null space of A, we look at the system $\mathbf{y}^T A = 0$. Recall the Gauss-Jordan elimination method from Section **??** that augments A with an $m \times m$ identity matrix, and performs row elimination on the augmented matrix.

$$[A \ I_{m\times m}] \stackrel{rref}{\Longrightarrow} [R \ E_{m\times m}]$$

The rref will consist of the reduced matrix augmented with the elimination matrix reproduced on its right.

Recap: The Four Subspaces and their Bases (contd.)

For the example case in 10, we apply the same elimination steps to obtain the matrix E below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{2,1}, E_{3,1}} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \xrightarrow{E_{3,2}} (E =) \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \tag{11}$$

Recap: The Four Subspaces and their Bases (contd.)

Writing down $EA = R$,

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

We observe that the last row of E specifies a linear combination of the rows of A that yields a zero vector (corresponding to the last row of R). This is the only vector that yields a zero row in R and is therefore the only element in the basis of the left null space of A, that is, $N(A^T)$. The dimension of $N(A^T)$ is 1.

Recap: The Four Subspaces and their Bases (contd.)

- As another example, consider the space $\mathcal{S}$ of vectors $\mathbf{v} \in \Re^3$ where $\mathbf{v} = [v_1 \;\; v_2 \;\; v_3]^T$ such that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. What is the dimension of this subspace?
- Note that this subspace is the right null space $N(A)$ of a 1×3 matrix $A = [1\ 1\ 1]$, since $A\mathbf{v} = 0$. The rank, $r = rank(A)$ is 1, implying that the dimension of the right null space is $n - r = 3 - 1 = 2$.
- One set of basis vectors for $\mathcal{S}$ is $[-1\ 1\ 0]$, $[-1\ 0\ 1]$. The column space $C(A)$ is $\Re^1$ with dimension 1. The left null space $N(A^T)$ is the singleton set $\{0\}$ and as expected, has a dimension of $m - r = 1 - 1 = 0$.

Recap: Matrix Spaces

We will extend the set of examples of vector spaces discussed in Section **??** with a new vector space, that of all $m \times n$ matrices with real entries, denoted by $\Re^{m \times n}$.

- It is easy to verify that the space of all matrices is closed under operations of addition and scalar multiplication. Additionally, there are interesting subspaces in the entire matrix space $\Re^{m \times n}$, *viz.*,
  - set $\mathcal{S}$ of all $n \times n$ symmetric matrices
  - set $\mathcal{U}$ of all $n \times n$ upper triangular matrices
  - set $\mathcal{L}$ of all $n \times n$ lower triangular matrices
  - set $\mathcal{D}$ of all $n \times n$ diagonal matrices
- Let $\mathcal{M} = \Re^{3 \times 3}$ be the space of all 3×3 matrices. The dimension of $\mathcal{M}$ is 9. Each element of this basis has a 1 in one of the 9 positions and the remaining entries as zeroes.
- Of these basis elements, three are symmetric (those having a 1 in any of the diagonal positions). These three matrices form the basis for the subspace of diagonal matrices.
- Six of the nine basis elements of M form the basis of $\mathcal{U}$ while six of them form the basis of $\mathcal{L}$.

Recap: Matrix Spaces (contd.)

- The intersection of any two matrix spaces is also a matrix space. For example, $\mathcal{S} \cap \mathcal{U}$ is $\mathcal{D}$, the set of diagonal matrices.
- However the union of any two matrix spaces need not be a matrix space. For example, $\mathcal{S} \cup \mathcal{U}$ is not a matrix space; the sum $S + U$, $S \in \mathcal{S}$, $U \in \mathcal{U}$ need not belong to $\mathcal{S} \cup \mathcal{U}$.
- We will discuss a special set comprising all linear combinations of the elements of union of two vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ (*i.e.*, $\mathcal{V}_1 \cup \mathcal{V}_2$), and denote this set by $\mathcal{V}_1 \oplus \mathcal{V}_2$. By definition, this set is a vector space. For example, $\mathcal{S} + \mathcal{U} = \mathcal{M}$, which is a vector space.

Recap: Matrix Spaces (contd.)

A property fundamental to many properties of matrices is the expression for a rank 1 matrix. A rank 1 matrix can be expressed as the product of a column vector with a row vector (the row vector forming a basis for the matrix). Thus, any rank 1 matrix X can be expressed as

$$X_{m\times n} = u^T v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \tag{13}$$

Recap: Matrix Spaces (contd.)

Let $\mathcal{M}_{m\times n}$ be the set of all $m \times n$ matrices. Is the subset of $\mathcal{M}_{m\times n}$ matrices with rank k, a subspace? For $k = 1$, this space is obviously not a vector space as is evident from the sum of rank 1 matrices, A^1 and B^1, which is not a rank 1 matrix. In fact, the subset of $\mathcal{M}_{m\times n}$ matrices with rank k is not a subspace.

$$A^1 + B^1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 2 \\ 2 & 2 & 1 \\ 4 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 3 \\ 4 & 6 & 2 \\ 5 & 6 & 3 \end{bmatrix} \tag{14}$$

Orthogonality and Projection

- Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal *iff*, their dot product (more generally, the inner product) is 0. In the eucledian space, the dot product of the two vectors is $x^T y$.
- The condition $x^T y = 0$ is equivalent to the pythagorous condition between the vectors $\mathbf{x}$ and $\mathbf{y}$ that form the perpendicular sides of a right triangle with the hypotenuse given by $\mathbf{x} + \mathbf{y}$. The *pythagorous condition* is $||\mathbf{x}||^2 + ||\mathbf{y}||^2 = ||\mathbf{x} + \mathbf{y}||^2$, where the norm is the eucledian norm, given by $||\mathbf{x}||^2 = \mathbf{x}^T\mathbf{x}$.
- This equivalence can be easily proved and is left to the reader as an exercise. By definition, the vector $\mathbf{0}$ is orthogonal to every other vector.

Orthogonality and Projection

We will extend the definition of orthogonality to subspaces; a subspace $\mathcal{U}$ is orthogonal to subspace $\mathcal{V}$ *iff*, every vector in $\mathcal{U}$ is orthogonal to every vector in $\mathcal{V}$. As an example:

Theorem

The row space $C(A^T)$ of an $m \times n$ matrix A is orthogonal to its right null space $N(A)$.

Proof: $A\mathbf{x} = \mathbf{0},\ \forall \mathbf{x} \in N(A)$. On the other hand, $\forall\ \mathbf{y} \in C(A^T),\ \exists\ z \in \Re^m,\ \mathbf{s}.t.,\ \mathbf{y} = A^T\mathbf{z}$. Therefore, $\forall\ \mathbf{y} \in C(A^T),\ \mathbf{x} \in N(A), \mathbf{y}^T\mathbf{x} = \mathbf{z}^T A\mathbf{x} = \mathbf{z}.\mathbf{0} = 0$. □

Orthogonality and Projection

Not only are $C(A^T)$ and the right null space $N(A)$ orthogonal to each other, but they are also *orthogonal complements* in $\Re^n$, that is, $N(A)$ contains all vectors that are orthogonal to some vector in $C(A^T)$.

Theorem

The null space of A and its row space are orthogonal complements.

Proof: We note, based on our discussion earlier that the dimensions of the row space and the (right) null space add up to n, which is the number of columns of A. For any vector $\mathbf{y} \in C(A^T)$, we have $\exists\ \mathbf{z} \in \Re^m,\ \mathrm{s}.t.,\ \mathbf{y} = A^T\mathbf{z}$. Suppose $\forall\ \mathbf{y} \in C(A^T),\ \mathbf{y}^T\mathbf{x} = 0$. That is, $\forall\ \mathbf{z} \in \Re^m,\ \mathbf{z}^T A\mathbf{x} = 0$. This is possible only if $A\mathbf{x} = \mathbf{0}$. Thus, necessarily, $\mathbf{x} \in N(A)$. □

Along similar lines, we could prove that the column space $C(A)$ and the left null space $N(A^T)$ are orthogonal complements in $\Re^m$.

Orthogonality and Projection

Based on preceeding theorem,we prove that there is a one-to-one mapping between the elements of row space and column space.

Theorem

If $\mathbf{x} \in C(A^T)$, $\mathbf{y} \in C(A^T)$ *and* $\mathbf{x} \neq \mathbf{y}$, *then,* $A\mathbf{x} \neq A\mathbf{y}$.

Proof: Note that $A\mathbf{x}$ and $A\mathbf{y}$ are both elements of $C(A)$. Next, observe that $\mathbf{x} - \mathbf{y} \in C(A^T)$, which by theorem 8, implies that $\mathbf{x} - \mathbf{y} \notin N(A)$. Therefore, $A\mathbf{x} - A\mathbf{y} \neq \mathbf{0}$ or in other words, $A\mathbf{x} \neq A\mathbf{y}$. □

Similarly, it can be proved that if $\mathbf{x} \in C(A)$, $\mathbf{y} \in C(A)$ and $\mathbf{x} \neq \mathbf{y}$, then, $A^T\mathbf{x} \neq A^T\mathbf{y}$. The two properties together imply a one-to-one mapping between the row and column spaces.

Projection Matrices

The projection of a vector $\mathbf{t}$ on a vector $\mathbf{s}$ is a vector $\mathbf{p} = c\mathbf{s}$, $c \in \Re$ (in the same direction as $\mathbf{s}$), such that $\mathbf{t} - c\mathbf{s}$ is orthogonal to $\mathbf{s}$. That is, $\mathbf{s}^T(\mathbf{t} - c\mathbf{s}) = 0$ or $\mathbf{s}^T\mathbf{t} = c\mathbf{s}^T\mathbf{s}$). Thus, the scaling factor c is given by $c = \frac{\mathbf{s}^T\mathbf{t}}{\mathbf{s}^T\mathbf{s}}$. The projection of the vector $\mathbf{t}$ on a vector $\mathbf{s}$ is then

$$\mathbf{p} = \mathbf{s}\frac{\mathbf{t}^T\mathbf{s}}{\mathbf{s}^T\mathbf{s}} \tag{15}$$

Using the associative property of matrix multiplication, the expression for $\mathbf{p}$ can be re-written as

$$\mathbf{p} = P\mathbf{t} \tag{16}$$

where, $P = \mathbf{s}\mathbf{s}^T\frac{1}{\mathbf{s}^T\mathbf{s}}$ is called the *projection matrix*.

Projection Matrices (contd.)

- The rank of the projection matrix is 1 (since it is a column mutiplied by a row).
- The projection matrix is symmetric and its column space is a line through s.
- For any $d \in \Re$, $P(d\mathbf{s}) = d\mathbf{s}$, that is, the projection of any vector in the direction of $\mathbf{s}$ is the same vector. Thus, $P^2 = P$.

Least Squares

- We earlier saw a method for solving the system $A\mathbf{x} = \mathbf{b}$ (A being an $m \times n$ matrix), when a solution exists. Howevever, a solution may not exist, especially when $m > n$, that is when the number of equations is greater than the number of variables.
- We also saw that the *rref* looks like $[I\ \mathbf{0}]^T$, where I is an $n \times n$ identity matrix. It could happen that the row reduction yields a zero submatrix in the lower part of A, but the corresponding elements in $\mathbf{b}$ are not zeroes.
- In other words, $\mathbf{b}$ may not be in the column space of A. In such cases, we are often interested in finding a 'best fit' for the system; a solution $\hat{x}$ that satisfies $A\mathbf{x} = \mathbf{b}$ as well as possible.

Projection Matrices (contd.)

- We define the best fit in terms of a vector $\mathbf{p}$ which is the projection of $\mathbf{b}$ onto $C(A)$ and solve $A\hat{\mathbf{x}} = \mathbf{p}$. We require that $\mathbf{b} - \mathbf{p}$ is orthogonal to $C(A)$, which means

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0} \tag{17}$$

- The vector $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$ is the error vector and is in $N(A^T)$. The equation (95) can be rewritten as

$$(A^TA)\hat{\mathbf{x}} = A^T\mathbf{b} \tag{18}$$

Projection Matrices (contd.)

A matrix that plays a key role in this problem is A^TA. It is an $n \times n$ symmetric matrix (since $(A^TA)^T = A^TA$). The right null space $N(A^TA)$ is the same as $N(A)^3$. It naturally follows that the ranks of A^TA and A are the same (since, the sum of the rank and dimension of null space equal n in either case). Thus, A^TA is invertible exactly if $N(A)$ has dimension 0, or equivalently, A is a full column rank.

Theorem

If A is a full column rank matrix (that is, its columns are independent), A^TA is invertible.

Proof: We will show that the null space of A^TA is $\{0\}$, which implies that the square matrix A^TA is full column (as well as row) rank is invertible. That is, if $A^TA\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. Note that if $A^TA\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^TA^TA\mathbf{x} = ||A\mathbf{x}|| = 0$ which implies that $A\mathbf{x} = \mathbf{0}$. Since the columns of A are linearly independent, its null space is $\mathbf{0}$ and therefore, $\mathbf{x} = \mathbf{0}$. □

Assuming that A is full column rank, the equation (18) can be rewritten as

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}. \tag{19}$$

Projection Matrices (contd.)

Therefore the expression for the projection $\mathbf{p}$ will be

$$\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b} \tag{20}$$

This expression is the n-dimensional equivalent of the one dimensional expression for projection in (92). The projection matrix in (20) is given by $P = A(A^TA)^{-1}A^T$.
We will list the solution for some special cases:

- If A is an $n \times n$ square invertible matrix, its column space is the entire $\Re^n$ and the projection matrix will turn out to be the identity matrix.
- Also, if b is in the column space $C(A)$, then $\mathbf{b} = A\mathbf{t}$ for some $t\ in \Re^n$ and consequently, $P\mathbf{b} = A(A^TA)^{-1}(A^TA)\mathbf{t} = A\mathbf{t} = \mathbf{b}$.
- On the other hand, if b is orthogonal to $C(A)$, it will lie in $N(A^T)$, and therefore, $A^T\mathbf{b} = 0$, implying that $\mathbf{p} = 0$.

Projection Matrices (contd.)

Another equivalent way of looking at the best fit solution $\hat{\mathbf{x}}$ is a solution that minimizes the square of the norm of the error vector

$$e(\hat{\mathbf{x}}) = ||A\mathbf{x} - \mathbf{b}||^2 \tag{21}$$

Setting $\frac{de(\hat{\mathbf{x}})}{d\mathbf{x}} = 0$, we get the same expression for $\hat{\mathbf{x}}$ as in (96). The solution in 96 is therefore often called the *least squares solution*. Thus, we saw two views of finding a best fit; first was the view of projecting into the column space while the second concerned itself with minimizing the norm squared of the error vector.

Projection Matrices (contd.)

We will take an example. Consider the data matrix A and the coefficient matrix $\mathbf{b}$ as in (22).

$$Ax = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \tag{22}$$

Projection Matrices (contd.)

The matrix A is full column rank and therefore A^TA will be invertible. The matrix A^TA is given as

$$A^TA = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

Substituting the value of A^TA in the system of equations (18), we get,

$$6\hat{x}_1 - 3\hat{x}_2 = 2 \quad (23)$$

$$-3\hat{x}_1 + 6\hat{x}_2 = 8 \quad (24)$$

The solution of which is, $x_1 = \frac{4}{5}$, $x_2 = \frac{26}{15}$.

Orthonormal Vectors

A collection of vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ is said to be orthonormal *iff* the following condition holds $\forall\ i, j$:

$$\mathbf{q}_i^T \mathbf{q}_j \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{25}$$

A large part of numerical linear algebra is built around working with orthonormal matrices, since they do not overflow or underflow. Let Q be a matrix comprising the columns $\mathbf{q}_1$ through $\mathbf{q}_n$. It can be easily shown that

$$Q^T Q = I_{n \times n}$$

Orthonormal Vectors (contd.)

When Q is square, $Q^{-1} = Q^T$. Some examples of matrices with orthonormal columns are:

$$Q_{rotation} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}, \; Q_{reflection} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{bmatrix},$$

$$Q_{Hadamard} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \; Q_{rect} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{26}$$

The matrix $Q_{rotation}$ when multiplied to a vector, rotates it by an angle θ, whereas $Q_{reflection}$ reflects the vector at an angle of $\theta/2$.

Orthonormal Vectors (contd.)

These matrices present standard varieties of linear transformation, but in general, premultiplication by an $m \times n$ matrix transforms from an input space in $\Re^m$ to an input space in $\Re^n$.

The matrix $Q_{Hadamard}$ is an orthonormal matrix consisting of only 1's and -1's. Matrices of this form exist only for specific dimensions such as 2, 4, 8, 16, *etc.*, and are called *Hadamard matrices*[4].

The matrix Q_{rect} is an example rectangular matrix whose columns are orthonormal.

[4] An exhaustive listing of different types of matrices can be found at `http://en.wikipedia.org/wiki/List_of_matrices`.

Orthonormal Vectors (contd.)

Suppose a matrix Q has orthonormal columns. What happens when we project any vector onto the column space of Q? Substituting $A = Q$ in (20), we get[5]:

$$\mathbf{p} = Q(Q^TQ)^{-1}Q^T\mathbf{b} = QQ^T\mathbf{b} \tag{27}$$

Making the same substitution in (96),

$$\hat{\mathbf{x}} = (A^TQ)^{-1}Q^T\mathbf{b} = Q^T\mathbf{b} \tag{28}$$

The i^{th} component of $\mathbf{x}$, is given by $x_i = q_i^T b$.
Let Q_1 be one orthonormal basis and Q_2 be another orthonormal basis for the same space. Let A be the coefficient matrix for a set of points represented using Q_1 and B be the coefficient matrix for the same set of points represented using Q_2. Then $Q_1A = Q_2B$, which implies that B can be computed as $B = Q_2^TQ_1A$. This gives us the formula for changing basis.

[5]Note that $Q^TQ = I$. However, $QQ^T = I$ only if Q is a square matrix.

Gram-Schmidt Orthonormalization

- The goal of the Gram-Schmidt orthonormalization process is to generate a set of orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, given a set of independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.
- The first step in this process is to generate a set of orthogonal vectors $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ from $\mathbf{a}_1, \mathbf{a}_2, \ldots, \bullet\, a_n$. To start with, $\mathbf{t}_1$ is chosen to be $\mathbf{a}_1$.
- Next, the vector $\mathbf{t}_2$ is obtained by removing the projection of $\mathbf{a}_2$ on $\mathbf{t}_1$, from $\mathbf{a}_2$, based on (92). That is,

$$\mathbf{t}_2 = \mathbf{a}_2 - \frac{1}{\mathbf{a}_1^T \mathbf{a}_1} \mathbf{a}_1 \mathbf{a}_1^T \mathbf{a}_2 \tag{29}$$

- This is carried out iteratively for $i = 1, 2, \ldots, n$, using the expression below:

$$\mathbf{t}_i = \mathbf{a}_i - \frac{1}{\mathbf{t}_1^T \mathbf{t}_1} \mathbf{t}_1 \mathbf{t}_1^T \mathbf{a}_i - \frac{1}{\mathbf{t}_2^T \mathbf{t}_2} \mathbf{t}_2 \mathbf{t}_2^T \mathbf{a}_i - \ldots - \frac{1}{\mathbf{t}_{i-1}^T \mathbf{t}_{i-1}} \mathbf{t}_{i-1} \mathbf{t}_{i-1}^T \mathbf{a}_i \tag{30}$$

Gram-Schmidt Orthonormalization (contd.)

- This iterative procedure gives us the orthogonal vectors $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$.
- Finally, the orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are obtained by the simple expression

$$\mathbf{q}_i = \frac{1}{||\mathbf{t}_i||}\mathbf{t}_i \tag{31}$$

- Let A be the matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and Q, the matrix with columns $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$.
- It can be proved that $C(V) = C(Q)$, that is, the matrices V and Q have the same column space. The vector a_i can be expressed as

$$\mathbf{a}_i = \sum_{k=1}^{n} (\mathbf{a}_i^T \mathbf{q}_k)\mathbf{q}_k \tag{32}$$

- The i^{th} column of A is a linear combination of the columns of Q, with the scalar coefficient $\mathbf{a}_i^T \mathbf{q}_k$ for the k^{th} column of Q.

Gram-Schmidt Orthonormalization (contd.)

- By the very construction procedure of the Gram-Schmidt orthonormalization process, $\mathbf{a}_i$ is orthogonal to $\mathbf{q}_k$ for all $k > i$. Therefore, (32) can be expressed more precisely as

$$\mathbf{a}_i = \sum_{k=1}^{i} (\mathbf{a}_i^T \mathbf{q}_k)\mathbf{q}_k \tag{33}$$

- Therefore, matrix A can be decomposed into the product of Q with a upper triangular matrix R; $A = QR$, with $R_{k,i} = \mathbf{a}_i^T \mathbf{q}_k$. Since $\mathbf{a}_i^T \mathbf{q}_k = 0,\ \forall\ k > i$, we can easily see that R is upper traingular.

End Recap: Basis and Dimensions from Linear Algebra wrt $< .,. >_E$ (Eucledian Inner Product) (For your homework)

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Proof:

- And here is how you can create a basis for $\mathcal{V}, < ., . >$:
  - If $\mathbf{u} = 0$ or $\mathbf{v} = 0$, then $< \mathbf{u}, \mathbf{v} >= 0$.
  - Assume $\mathbf{u}, \mathbf{v} \neq 0$. Let $\mathbf{z} = \mathbf{u} - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}$.
  - By linearity of inner product in first argument, we have: $< \mathbf{z}, \mathbf{v} >=< \mathbf{u} - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}, \mathbf{v} >=< \mathbf{u}, \mathbf{v} > - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>} < \mathbf{v}, \mathbf{v} >= 0$
  - Therefore, $< \mathbf{u}, \mathbf{u} >=< \mathbf{z} + \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}, \mathbf{z} + \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v} >=< \mathbf{z}, \mathbf{z} > + (\frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>})^2 < \mathbf{v}, \mathbf{v} > + 0$
  - So $< \mathbf{u}, \mathbf{u} > \geq \frac{|<\mathbf{u},\mathbf{v}>|^2}{<\mathbf{v},\mathbf{v}>}$

Compact representation of Inner Product Space

- Let the linear subspace $S \subseteq V$ be associated with an inner product $< .,. >$
- Let $B = basis(S)$ with respect to the arbitrary inner product $< .,. >$ (extending results from the eucledian inner product)
- Let $dim(V) = n$, and $dim(S) = m \leq n$.
- Define $S^{\perp}$; the orthogonal complement ($S^{\perp} \in V$) of S as:
  $S^{\perp} = \{v \in V \mid < v, u >= 0 \; \forall \; u \in S \}$
  This implies:-
    - Both S and $S^{\perp}$ are linear subspaces of V.
    - $S \cap S^{\perp} = \{0\}$, $dim(S) + dim(S^{\perp}) = n$
    - $(S^{\perp})^{\perp} = S$.
    - If $B^{\perp}$ is the basis for $S^{\perp}$, then $B \cup B^{\perp}$ is the basis for V.
    - $S = \{\mathbf{v} \in V | < \mathbf{v}, \mathbf{u} >= 0, \; \forall \; \mathbf{u} \in B^{\perp} \}$
    - $S^{\perp} = \{\mathbf{v} \in V \mid < \mathbf{v}, \mathbf{u} >= 0 \; \forall \; \mathbf{u} \in B \}$

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Motivation:

- Consider the following inner product on $\Re^2$: For any $\mathbf{x}, \mathbf{y} \in \Re^2$, let $< \mathbf{x}, \mathbf{y} >= 2x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. It can be easily verified that this in an inner product (by checking for linearity, symmetry and positive definiteness by expressing it as a sum of squares).
- This inner product is certainly different from the conventional (Eucledian) dot product $< \mathbf{x}, \mathbf{y} >_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $< \mathbf{x}, \mathbf{y} >$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Motivation:

- Consider the following inner product on $\Re^2$: For any $\mathbf{x}, \mathbf{y} \in \Re^2$, let $< \mathbf{x}, \mathbf{y} >= 2x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. It can be easily verified that this in an inner product (by checking for linearity, symmetry and positive definiteness by expressing it as a sum of squares).
- This inner product is certainly different from the conventional (Eucledian) dot product $< \mathbf{x}, \mathbf{y} >_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $< \mathbf{x}, \mathbf{y} >$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

In $\Re^n$, it can be proved that for any inner product vector space $(\mathcal{V}, < .,. >)$, the inner product $< .,. >$ (including the Eucledian one) can be represented as

$$< \mathbf{u}, \mathbf{v} >= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j < \mathbf{e}_i, \mathbf{e}_j >= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}^T E \mathbf{b} =< \mathbf{a}^T, \mathbf{b} >_E$$

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Proof:

- In $\Re^n$, it can be proved that for any inner product vector space $(\mathcal{V}, < .,. >)$, the inner product $< .,. >$ (including the Eucledian one) can be represented as $< \mathbf{u}, \mathbf{v} >= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j < \mathbf{e}_i, \mathbf{e}_j >= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}^T E \mathbf{b} =< \mathbf{a}^T, \mathbf{b} >_E$
  - Here, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a basis for the inner product vector space.
  - The inner product $< .,. >_E$ is the eucledian inner product. That is, $< .,. >_E = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j$.

    The (positive definite) matrix E is defined as

$$E = \begin{bmatrix} < \mathbf{e}_1, \mathbf{e}_1 > & < \mathbf{e}_1, \mathbf{e}_2 > & \ldots\ldots & < \mathbf{e}_1, \mathbf{e}_n > \\ & . & & \\ < \mathbf{e}_n, \mathbf{e}_1 > & < \mathbf{e}_n, \mathbf{e}_2 > & \ldots\ldots & < \mathbf{e}_n, \mathbf{e}_n > \end{bmatrix} \quad (34)$$

  - Note that in any $\Re^n$, any inner product vector space $(\mathcal{V}, < .,. >)$ will have a basis of size at most n.

HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Thus, any inner product $< ., . >$ in $\Re^n$ can be expressed as a Eucledian inner product $< ., . >_E$, with possible rotation using a matrix R where $E = RR^T$ is a positive definite matrix[6]

[6] Recall from slides 25 to 27 that $\mathbf{x}^P\mathbf{x}$ is a norm if P is positive definite

Dual Representation: Explained with Analogy

If $S \subseteq \Re^n$ and $\{a_1, a_2, ..., a_K\}$ is finite spanning set in $S^\perp$, then:-

- $S = (S^\perp)^\perp = \{\mathbf{x} | a_i \mathbf{x} = 0; i = 1, ..., k\}$
- A dual representation of linear subspace S(in $\Re^n$): $\{\mathbf{x} | A\mathbf{x} = 0;\ a_i^T$ is the i^{th} row of $A\}$

Dual Representations of Affine Sets

Recall affine sets(say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A(= V$ shifted by $\mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n$ is fixed and $\mathbf{v} \in V\ \}$.
- For some P with rank = n - dim(V) and b, A is affine iff:
  $A = \{\mathbf{x} | P\mathbf{x} = b\}$ i.e. solution set of linear equations represented by $P\mathbf{x} = b$.
- Example: In 3-d if P has rank 1, we will get either a plane as solution or no solution. If P has rank 2, we can get a plane, a line or no solution.
- Thus hyperplanes are affine spaces of dimension $n-1$ with $P\mathbf{x} = \mathbf{b}$ given by $p^T\mathbf{x} = \mathbf{b}$.
  For 3-d we have P
  We will soon see the duality of convex cones, and in general convex sets.

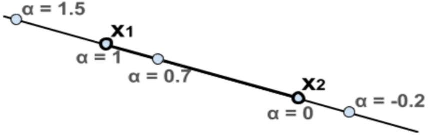
Convex Sets

Convex sets

- affine and convex sets.
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

- **In 2D, a line** through any two distinct points $\mathbf{x}_1, \mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.



  $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ where $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$
- In general, A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta\mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A (= V \text{ shifted by } \mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n \text{ is fixed and } \mathbf{v} \in V\ \}$.
- For some P with rank $= n - dim(V)$ and $\mathbf{b}$, A is affine iff:
  $A = \{\mathbf{x} | P\mathbf{x} = b\}$ i.e. solution set of linear equations represented by $P\mathbf{x} = b$.
  - No Solution: $\mathbf{x} = \phi$. Is that affine?
  - Unique Solution: $\mathbf{x}$ is a point.
  - Infinitely Many Solutions: $\mathbf{x}$ is a line, or a plane, etc.

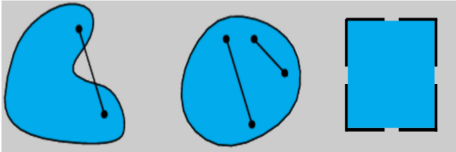
(conversely every affine set can be expressed as solution set of system of linear equations)

Convex set

- **In 2D, a line segment** between distinct points $\mathbf{x}_1$,$\mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.

$$\begin{aligned} \mathsf{x} = & \quad \alpha x_1 + \beta x_2 \\ \text{where} & \quad \alpha + \beta = 1, 0 \leq \alpha \leq 1 (\textit{also}, 0 \leq \beta \leq 1). \end{aligned}$$

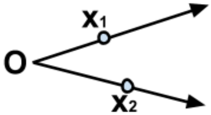
- **Convex set** : $\mathbf{x}_1, \mathbf{x}_2 \in C, 0 \leq \alpha \leq 1 \Rightarrow \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in C$



  - Convex set is connected. Convex set can but not necessarily contains 'O'

Is every affine set convex? Is the reverse true?

Cone, conic combination and convex cone

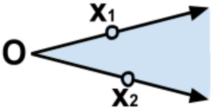


- **Cone** A set C is a cone if $\forall \mathbf{x} \in C$, $\alpha\mathbf{x} \in C$ for $\alpha \geq 0$.
- **Conic (nonnegative) combination** of points $\mathbf{x}_1, \mathbf{x}_2$ is any point $\mathbf{x}$ of the form

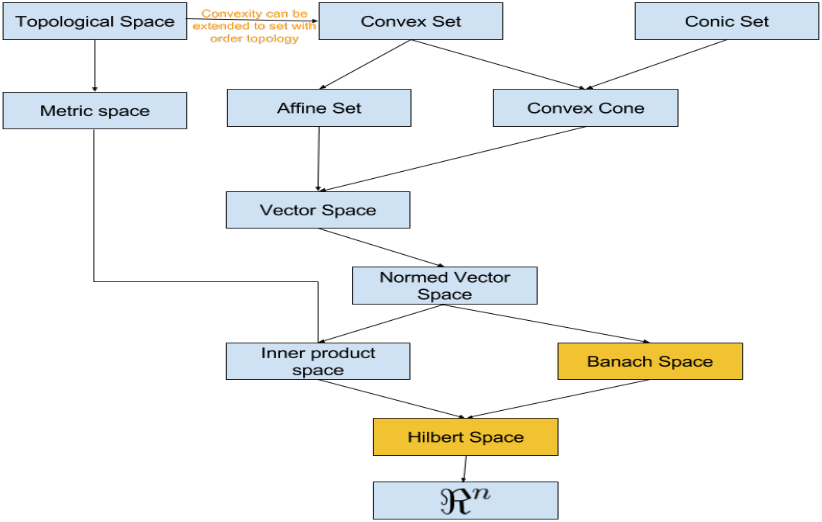
$$\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$$

$$\text{with} \quad \alpha, \beta \geq 0.$$

  Example : Diagonal vector of a parallelogram is a conic combination of the two vectors (points) $\mathbf{x}_1$ and $\mathbf{x}_2$ forming the sides of the parallelogram.
- **Convex cone:** The set that contains all conic combinations of points in the set.



Homework: Structure of Mathematical Spaces Discussed (arrow means 'instance')



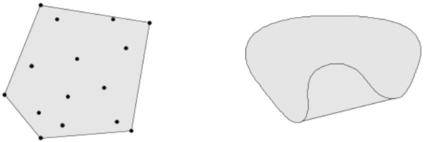
Convex combination and convex hull

- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- **Convex hull or conv(S)** is the set of all convex combinations of point in the set S.

  - 

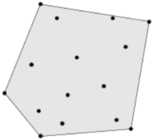
- Should S be always convex?
- What about the convexity of conv(S)?

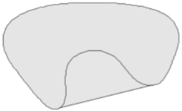
Convex combination and convex hull

- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\begin{aligned} \mathbf{x} = \quad & \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}) \\ \text{with} \quad & \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0. \end{aligned}$$

- **Convex hull or conv(S)** is the set of all convex combinations of point in the set S.





- Should S be always convex? No.
- What about the convexity of conv(S)? It's always convex.

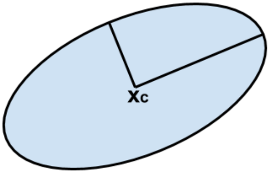
More Convex Sets (illustrated in $\Re^n$)

More Convex Sets (illustrated in $\Re^n$)

- Euclidean balls and ellipsoids.
- Norm balls and norm cones.
- Compact representation of vector space.
- Dual Representation.
- Different Representations of Affine Sets

Euclidean balls and ellipsoids

- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \leq 1\}$
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\}$, where P $\in S^n_{++}$ i.e. P is SPD matrix.
  - Other representation: $\{\mathbf{x}_c + \mathrm{A}\ \mathrm{u} \mid \|u\|_2 \leq 1\}$ with A square and non-singular(i.e. A^{-1} exists).



Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) =$

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$,

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\underset{s \in S}{sup}\ f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_{j} \sum\limits_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$,

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$,

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s\in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum\limits_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$, $M_N(A) = \max\limits_i \sum\limits_{j=1}^{m} |a_{ij}|$

$N = \|.\|_1, \; M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation:

$$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \le \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \le \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then

$$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_{j} \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x}$

$$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then

$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then $\|\mathbf{x}\|_1 = 1$ and $\|A\mathbf{x}\|_1 = C$
5. Thus, there exists $\mathbf{x} = [0, 0..1, 0...0]$ for which the inequalities in steps (2) and (3) become equalities! That is,

$N = \|.\|_1$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then $\|\mathbf{x}\|_1 = 1$ and $\|A\mathbf{x}\|_1 = C$
5. Thus, there exists $\mathbf{x} = [0, 0..1, 0...0]$ for which the inequalities in steps (2) and (3) become equalities! That is,

$$M_N(A) = \|A\mathbf{x}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[8]):

[8] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[8]): $A^TA \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition,

[8] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[8]): $A^TA \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition, there exists orthonormal U with column vectors $\mathbf{u}_i$ and diagonal matrix Σ of non-negative eigenvalues σ_i of A^TA such that $A^TA = U^T\Sigma U$ with $(A^TA)\mathbf{u}_i = \sigma_i\mathbf{u}_i$
4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} =$

[8] `https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf`

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[8]): $A^TA \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition, there exists orthonormal U with column vectors $\mathbf{u}_i$ and diagonal matrix Σ of non-negative eigenvalues σ_i of A^TA such that $A^TA = U^T\Sigma U$ with $(A^TA)\mathbf{u}_i = \sigma_i\mathbf{u}_i$
4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$
6. Then, $\|\mathbf{x}\|_2 = \sqrt{\sum_i \alpha_i^2}$ and $\|A\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T(A^TA\mathbf{x})} =$

[8] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[8]): $A^TA \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition, there exists orthonormal U with column vectors $\mathbf{u}_i$ and diagonal matrix Σ of non-negative eigenvalues σ_i of A^TA such that $A^TA = U^T\Sigma U$ with $(A^TA)\mathbf{u}_i = \sigma_i\mathbf{u}_i$
4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} = \sum_{i=1}^{n} \alpha_i\mathbf{u}_i$
6. Then, $\|\mathbf{x}\|_2 = \sqrt{\sum_i \alpha_i^2}$ and $\|A\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T(A^TA\mathbf{x})} = \sqrt{(\sum_{i=1}^{n}\alpha_i\mathbf{u}_i)^T(\sum_{i=1}^{n}\sigma_i\alpha_i\mathbf{u}_i)}$.
7. If $\alpha_1 = 1$ and $\alpha_j = 0$ for all $j \neq 1$, the maximum value in (7) will be attained. Thus, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of A^TA

[8] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

Norm balls: Summary

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set.
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$, $M_N(A) = \max_i \sum_{j=1}^{m} |a_{ij}|$
- Matrix norm with an inner product: $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{trace(A^T A)}$ is the Frobenius norm.

HW: Dual Representation

If vector space $V \subseteq \Re^n$ and $\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_K\}$ is finite spanning set in $V^\perp$, then:-

- $V = (V^\perp)^\perp = \{\mathbf{x} | \mathbf{q}_i^T \mathbf{x} = 0; i = 1, ..., K\}$, where $K = dim(V)$
- A dual representation of vector subspace V (in $\Re^n$): $\{\mathbf{x} | Q\mathbf{x} = 0;\ \mathbf{q}_i^T$ is the i^{th} row of $Q\}$
- What about dual representations for Affine Sets, Convex Sets, Convex Cones, etc?

HW: Dual Representations of Affine Sets

Recall affine sets(say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A(= V \text{ shifted by } \mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n \text{ is fixed and } \mathbf{v} \in V\ \}$.
- Procedure: Let $\mathbf{u}$ be some element in the affine set A. Then $V(= A \text{ shifted by } -\mathbf{u}) = \{\ \mathbf{v} - \mathbf{u} | \mathbf{v} \in A\ \}$ is a vector space which has a dual representation $\{\mathbf{x} | Q\mathbf{x} = 0\}$
- The dual representation for A is therefore

HW: Dual Representations of Affine Sets

Recall affine sets(say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1 - \theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A(= V$ shifted by $\mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n$ is fixed and $\mathbf{v} \in V\ \}$.
- Procedure: Let $\mathbf{u}$ be some element in the affine set A. Then $V(= A$ shifted by $-\mathbf{u}) = \{\ \mathbf{v} - \mathbf{u} | \mathbf{v} \in A\ \}$ is a vector space which has a dual representation $\{\mathbf{x} | Q\mathbf{x} = 0\}$
- The dual representation for A is therefore $\{\mathbf{x} | Q\mathbf{x} = Q\mathbf{u}\}$

HW: Dual Representations of Affine Sets

- For some Q with $rank = n - dim(V)$ and $\mathbf{u}$, A is affine iff:
  $A = \{\mathbf{x} | Q\mathbf{x} = Q\mathbf{u}\}$ i.e. solution set of linear equations represented by $Q\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = Q\mathbf{u}$.
- Example: In 3-d if Q has rank 1, we will get either a plane as solution or no solution. If Q has rank 2, we can get a plane, a line or no solution.
- Thus hyperplanes are affine spaces of dimension $n - 1$ with $Q\mathbf{x} = \mathbf{b}$ given by $p^T\mathbf{x} = \mathbf{b}$. We will soon see the duality of convex cones, and in general convex sets.

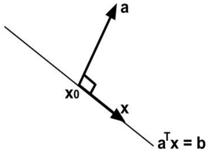
Examples of Convex Cones

More on Convex Sets and Cones

- Half-spaces as cones (induced by hyperplanes)
- Norm Cones
- Positive Semi-definite cone.
- Positive Semi-definite cone: Example and Notes.
- Convexity Preserving Operations on Sets

Hyperplanes and halfspaces.

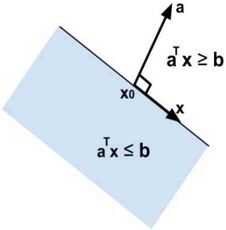
Hyperplane: Set of the form $\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\}$ $(\mathbf{a} \neq 0)$



- where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$
- Alternatively: $\{\mathbf{x}|(\mathbf{x} - \mathbf{x}_0) \perp \mathbf{a}\}$, where $\mathbf{a}$ is normal and $\mathbf{x}_0 \in H$

Hyperplanes and halfspaces.

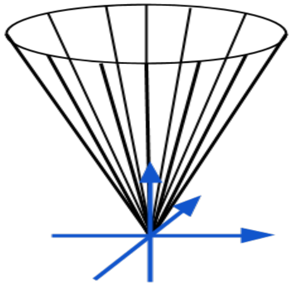
halfspace: Set of the form $\{\mathbf{x}|\mathbf{a}^T\mathbf{x} \leq \mathbf{b}\}$ $(\mathbf{a} \neq 0)$



- where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$

Norm cones

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius r**: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$.
- **Norm cone:** A **set** of form: $\{(\mathbf{x}, t) \in \Re^{n+1} | \|\mathbf{x}\| \leq t\}$.
  - Norm balls and cones are convex.
  - Euclidean norm cone is called-second order cone. If $\mathbf{x} \in \Re^2$, it is shown in $\Re^3$ as:-



Positive semidefinite cone: **Primal Description**

Notation

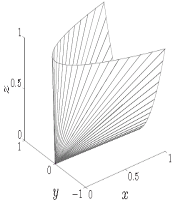
- S^n is set of symmetric $n \times n$ matrices.
- $S^n_+ = \{X \in S^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices.
    - $X \in S^n_+ \iff z^T X z \geq 0$ for all z
    - S^n_+ is a convex cone.
- $S^n_{++} = \{X \in S^n \mid X \succ 0\}$: positive definite $n \times n$ matrices.

Positive semidefinite cone: **Primal Description**

Consider a positive semi-definite matrix S in $\Re^2$. Then S must be of the form

$$S = \begin{bmatrix} \mathbf{x} & y \\ y & z \end{bmatrix} \quad (35)$$

We can represent the space of matrices $\mathcal{S}_+^2$ of the form $S \in \mathcal{S}_+^2$ as a three dimensional space with non-negative $\mathbf{x}$, y and z coordinates and a non-negative determinant. This space



corresponds to a cone as shown in the Figure above.

Positive semidefinite cone: **Dual Description**

Instead of all vectors $\mathbf{z} \in \Re^n$, we can, without loss of generality, only require the above inequality to hold for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1$.

1. $S_+^n = \{A \in S^n | A \succeq 0\} = \{A \in S^n | \mathbf{y}^T A \mathbf{y} \succeq 0, \forall \|\mathbf{y}\|_2 = 1\}$
2. So, $S_+^n = \cap_{\|\mathbf{y}\|=1} \{A \in S| < \mathbf{y}^T \mathbf{y}, A > \succeq 0\}$
3. $\mathbf{y}^T A \mathbf{y} = \sum_i \sum_j y_i a_{ij} y_j = \sum_i \sum_j (y_i y_j) a_{ij} = < \mathbf{y}\mathbf{y}^T, A > = tr((\mathbf{y}\mathbf{y}^T)^T A) = tr(\mathbf{y}\mathbf{y}^T A)$
    - One parametrization for $\mathbf{y}$ such that $\|\mathbf{y}\|_2 = 1$ is

$$\mathbf{y} = \begin{bmatrix} Cos(\theta) \\ Sin(\theta) \end{bmatrix} \quad (36)$$

$$\mathbf{y}\mathbf{y}^T = \begin{bmatrix} Cos^2(\theta) & Cos(\theta)Sin(\theta) \\ Cos(\theta)Sin(\theta) & Sin^2(\theta) \end{bmatrix} \quad (37)$$

    - Assignment 1: Plot a finite # of halfspaces parameterized by (θ).

Positive semidefinite cone: **Dual Description**

1. S_+^n = intersection of infinite # of half spaces belonging to $R^{n(n+1)/2}$ [Dual Representation]
    1. Cone boundary consists of all singular p.s.d. matrices having at-least one 0 eigenvalue.
    2. Origin = O = matrix with all 0 eigenvalues.
    3. Interior consists of all full rank matrices A (rank $A = m$) i.e. $A \succ 0$.

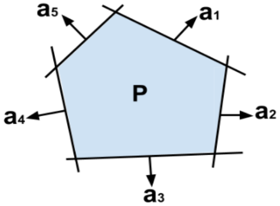
HW: $N = \|.\|_\infty$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})} = \sup_{\|\mathbf{x}\|=1} N(A\mathbf{x})$

1. If $N(\mathbf{x}) = \max_i |x_i|$ then $N(A\mathbf{x}) = \max_i |\sum_{j=1}^{m} a_{ij}x_j| \leq \max_i \sum_{j=1}^{m} |a_{ij}||x_j| \leq\leq \max_i \sum_{j=1}^{m} |a_{ij}|$ where the last inequality is attained by considering a $\mathbf{x} = [1, 1..1, 1...1]$ which has 1 in all positions. Then $\|\mathbf{x}\|_\infty = 1$ and for such an $\mathbf{x}$, the upper bounded on the supremum in indeed attained.
2. Therefore, it must be that $\|A\mathbf{x}\|_1 = \max_i \sum_{j=1}^{m} |a_{ij}|$ (the maximum absolute row sum)
3. That is,

$$M_N(A) = \|A\mathbf{x}\|_1 = \max_i \sum_{j=1}^{m} |a_{ij}|$$

Convex Polyhedron

- Solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$
  - $A \in \Re^{m \times n}$
  - $C \in \Re^{p \times n}$
  - $\preceq$ is component wise inequality



- This is a **Dual or *H* Description**: Intersection of finite number of half-spaces and hyperplanes.
- **Primal or *V* Description**: Can you define convex polyhedra in terms of convex hull?
  1. Convex hull of finite # of points $\Rightarrow$ **Convex Polytope**
  2. Conic hull of finite # of points $\Rightarrow$ **Polyhedral Cone**
  3. Convex hull of $n+1$ affinely independent points $\Rightarrow$ **Simplex**

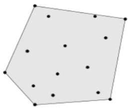
Convex combinations and Convex Hull

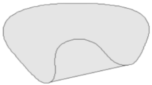
- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- Equivalent Definition of Convex Set: C is convex iff it is closed under generalized convex combinations.
- **Convex hull or** $conv(S)$ is the set of all convex combinations of points in the set S.
- $conv(S) =$ The smallest convex set that contains S. S may not be convex but $conv(S)$ is.
  - Prove by contradiction that if a point lies in another smallest convex set , and not in $conv(S)$, then it must be in $conv(S)$.





- The idea of convex combination can be generalized to include infinite sums, integrals, and, in most general form, probability distributions.

Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Balls in $\Re^n$]: Consider a point $\mathbf{x} \in \Re^n$. Then the closed ball around $\mathbf{x}$ of radius ϵ is

$$\mathcal{B}[\mathbf{x}, \epsilon] = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| \leq \epsilon\right\}$$

Likewise, the open ball around $\mathbf{x}$ of radius ϵ is defined as

$$\mathcal{B}(\mathbf{x}, \epsilon) = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| < \epsilon\right\}$$

For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when there exists an $\epsilon > 0$ such that $\mathcal{S} \subseteq \mathcal{B}[0, \epsilon]$.

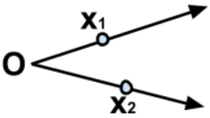
In other words, a set $\mathcal{S} \subseteq \Re^n$ is bounded means that there exists a number $\epsilon > 0$ such that for all $\mathbf{x} \in \mathcal{S}$, $||\mathbf{x}|| \leq \epsilon$.

Convex Polytope: **Primal** and **Dual** Descriptions

Dual or H Description: A Convex Polytope P is a **Bounded Convex Polyhedron**. That is, is solution set of finitely many inequalities or equalities: $P = \{\mathbf{x} | A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}\}$ where $A \in \Re^{m \times n}$, $C \in \Re^{p \times n}$ such that P is also bounded.

Primal or V Description : If $\exists$ $S \subset$ P s.t. $|S|$ is finite and $P = conv(S)$, then P is a **Convex Polytope**.

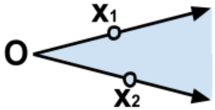
Conic combinations and Conic Hull



- Recap **Cone**: A set C is a cone if $\forall \mathbf{x} \in C$, $\theta \mathbf{x} \in C$ for $\theta \geq 0$.
- **Conic (nonnegative) Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2 + ... + \theta_k \mathbf{x}_k$$
$$\text{with} \quad \theta_i \geq 0.$$

- **Conic hull or** *conic*(S)**:** The set that contains all conic combinations of points in set S.



- *conic*(S) = Smallest conic set that contains S.

Polyhedral Cone: **Primal** and **Dual** Descriptions

Dual or *H* Description : A Polyhedral Cone P is a Convex Polyhedron with $\mathbf{b} = 0$. That is, $\{\mathbf{x} | A\mathbf{x} \succeq 0\}$ where $A \in \Re^{m \times n}$ and $\succeq$ is component wise inequality.

Primal of *V* Description : If $\exists\ S \subset$ P s.t. $|S|$ is finite and $P = cone(S)$, then P is a **Polyhedral Cone**.

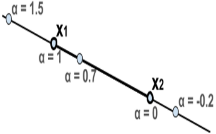
Affine combinations, Affine hull and Dimension of a set S

- **Affine Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k$$

$$\text{with} \quad \sum_i \theta_i = 1$$

- **Affine hull or aff(S):** The set that contains all affine combinations of points in set S = Smallest affine set that contains S.



-
- Dimension of a set S = dimension of $aff(S)$ = dimension of vector space V such that $aff(S) = \mathbf{v} + V$ for some $\mathbf{v} \in aff(S)$.
- $S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$ is set of $n+1$ *affinely independent* points if $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_{n+1} - \mathbf{x}_0\}$ are linearly independent.

Simplex (plural: simplexes) Polytope: **Primal** and **Dual** Descriptions

Dual or *H* Description: An n Simplex S is a convex Polytope with of affine dimension n and having $n+1$ corners. That is, is solution set of finitely many inequalities or equalities: $S = \{\mathbf{x} | A\mathbf{x} \preceq \mathbf{b} \ , \ C\mathbf{x} = \mathbf{d}\}$ where $A \in \Re^{m \times n}$, $C \in \Re^{p \times n}$ such that S with affine dimension n and having $n+1$ corners.

Primal or *V* Description: Convex hull of $n+1$ affinely independent points. Specifically, let $S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$ be a set of $n+1$ affinely independent points, then an n-dimensional simplex is $conv(S)$.
Simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

Convexity preserving operations

In practice if you want to establish the convexity of a set $\mathcal{C}$, you could either

1. prove it from first principles, *i.e.*, using the definition of convexity or
2. prove that $\mathcal{C}$ can be built from simpler convex sets through some basic operations which preserve convexity.

Some of the important operations that preserve complexity are:

1. Intersection
2. Affine Transform
3. Perspective and Linear Fractional Function

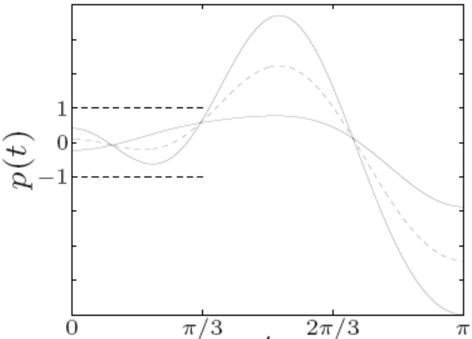
Closure under Intersection

The intersection of any number of convex sets is convex. Consider the set $\mathcal{S}$:

$$\mathcal{S} = \left\{ \mathbf{x} \in \Re^n \mid |p(t)| \leq 1 \ \textit{for} \ |t| \leq \frac{\pi}{3} \right\} \tag{38}$$

where

$$p(t) = x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \tag{39}$$



Closure under Intersection (contd.)

Any value of t that satisfies $|p(t)| \leq 1$, defines two regions, *viz.*,

$$\Re^{\leq}(t) = \left\{\mathbf{x} \mid x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \leq 1\right\}$$

and

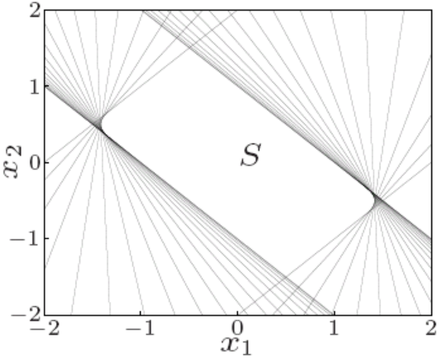
$$\Re^{\geq}(t) = \left\{\mathbf{x} \mid x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \geq -1\right\}$$

Each of the these regions is convex and for a given value of t, the set of points that may lie in $\mathcal{S}$ is given by $\Re(t) = \Re^{\leq}(t) \cap \Re^{\geq}(t)$

Closure under Intersection (contd.)

$\Re(t)$ is also convex. However, not all the points in $\Re(t)$ lie in $\mathcal{S}$, since the points that lie in $\mathcal{S}$ satisfy the inequalities for every value of t. Thus, $\mathcal{S}$ can be given as:

$$\mathcal{S} = \cap_{|t| \leq \frac{\pi}{3}} \Re(t)$$



Closure under Affine transform

An affine transform is one that preserves

- Collinearity between points, *i.e.*, three points which lie on a line continue to be collinear after the transformation.
- Ratios of distances along a line, *i.e.*, for distinct colinear points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, $\frac{||\mathbf{p}_2 - \mathbf{p}_1||}{||\mathbf{p}_3 - \mathbf{p}_2||}$ is preserved.

An affine transformation or affine map between two vector spaces $f : \Re^n \rightarrow \Re^m$ consists of a linear transformation followed by a translation:

$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$$

where $A \in \Re^{n \times m}$ and $\mathbf{b} \in \Re^m$.

Closure under Affine transform (contd.)

In the finite-dimensional case each affine transformation is given by a matrix A and a vector $\mathbf{b}$. The image and pre-image of convex sets under an affine transformation defined as

$$f(\mathbf{x}) = \sum_{i}^{n} x_i a_i + b$$

yield convex sets[9]. Here a_i is the i^{th} row of A. The following are examples of convex sets that are either images or inverse images of convex sets under affine transformations:

1. the solution set of linear matrix inequality $(A_i, B \in \mathcal{S}^m)$

$$\left\{\mathbf{x} \in \Re^n \mid x_1 A_1 + \ldots + x_n A_n \preceq B\right\}$$

   is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[10]. This set is the inverse image under an affine mapping of the

[9] Exercise: Prove.

[10] The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}^n_+$.

Closure under Affine transform (contd.)

In the finite-dimensional case each affine transformation is given by a matrix A and a vector $\mathbf{b}$. The image and pre-image of convex sets under an affine transformation defined as

$$f(\mathbf{x}) = \sum_{i}^{n} x_i a_i + b$$

yield convex sets[9]. Here a_i is the i^{th} row of A. The following are examples of convex sets that are either images or inverse images of convex sets under affine transformations:

1. the solution set of linear matrix inequality $(A_i, B \in \mathcal{S}^m)$

$$\left\{\mathbf{x} \in \Re^n \mid x_1A_1 + \ldots + x_nA_n \preceq B\right\}$$

   is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[10]. This set is the inverse image under an affine mapping of the spositive semi-definite cone. That is, $f^{-1}(cone) = \left\{\mathbf{x} \in \Re^n \mid B - (x_1A_1 + \ldots + x_nA_n) \in \mathcal{S}_+^m\right\} = \left\{\mathbf{x} \in \Re^n \mid B \geq (x_1A_1 + \ldots + x_nA_n)\right\}$.

[9]Exercise: Prove.

[10]The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}_+^n$.

Closure under Affine transform (contd.)

2. hyperbolic cone ($P \in \mathcal{S}^n_+$), which is the inverse image of the

Closure under Affine transform (contd.)

2. hyperbolic cone ($P \in \mathcal{S}_+^n$), which is the inverse image of the norm cone $\mathcal{C}_{m+1} = \{(\mathbf{z}, u) \,|\, ||\mathbf{z}|| \leq u, u \geq 0, \mathbf{z} \in \Re^m\} = \left\{(\mathbf{z}, u) | \mathbf{z}^T\mathbf{z} - u^2 \leq 0,\ u \geq 0, \mathbf{z} \in \Re^m\right\}$ is a convex set. The inverse image is given by $f^{-1}(\mathcal{C}_{m+1}) = \left\{\mathbf{x} \in \Re^n \mid \left(A\mathbf{x}, \mathbf{c}^T\mathbf{x}\right) \in \mathcal{C}_{m+1}\right\} = \left\{\mathbf{x} \in \Re^n | \mathbf{x}^T A^T A\mathbf{x} - (\mathbf{c}^T\mathbf{x})^2 \leq 0\right\}$. Setting, $P = A^T A$, we get the equation of the hyperbolic cone:

$$\left\{\mathbf{x} \mid \mathbf{x}^T P\mathbf{x} \leq (\mathbf{c}^T\mathbf{x})^2, \mathbf{c}^T\mathbf{x} \geq 0\right\}$$

Closure under Perspective and linear-fractional functions

The perspective function $P : \Re^{n+1} \rightarrow \Re^n$ is defined as follows:

$$P : \Re^{n+1} \rightarrow \Re^n \text{ such that} \qquad\qquad (40)$$
$$P(x,t) = x/t \qquad \textbf{dom } P = \{(x,t) \mid t > 0\}$$

The linear-fractional function f is a generalization of the perspective function and is defined as: $\Re^n \rightarrow \Re^m$:

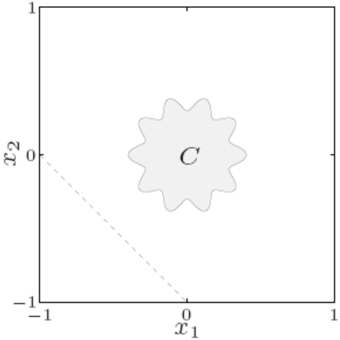
$$f : \Re^n \rightarrow \Re^m \text{ such that} \qquad\qquad (41)$$
$$f(\mathbf{x}) = \frac{A\mathbf{x}+\mathbf{b}}{\mathbf{c}^T\mathbf{x}+d} \qquad \textbf{dom } f = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} + d > 0\}$$

The images and inverse images of convex sets under perspective and linear-fractional functions are convex[11].

[11] Exercise: Prove.

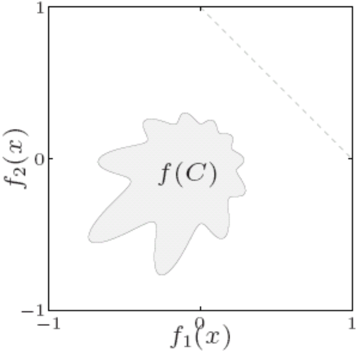
Closure under Perspective and linear-fractional functions (contd)

The Figure below shows an example set.



Closure under Perspective and linear-fractional functions (contd)

Consider the linear-fractional function $f = \frac{1}{x_1+x_2+1}x$. The following Figure shows the image of the set (from the prevous slide) under the linear-fractional function f.



Convex Functions, Epigraphs, Sublevel sets, Separating and Supporting Hyperplane Theorems and required tools

Convex Functions: Extending Slopeless Definition from $\Re :\rightarrow \Re$

Convex Functions: Extending Slopeless Definition from $\Re :\rightarrow \Re$

- A function $f : \mathcal{D} \rightarrow \Re$ is **convex** if

Convex Functions: Extending Slopeless Definition from $\Re :\rightarrow \Re$

- A function $f: \mathcal{D} \rightarrow \Re$ is **convex** if $\mathcal{D}$ is a convex set and

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{42}$$

- A function $f: \mathcal{D} \rightarrow \Re$ is **strictly convex** if $\mathcal{D}$ is convex and

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) < \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{43}$$

- A function $f: \mathcal{D} \rightarrow \Re$ is **strongly convex** if $\mathcal{D}$ is convex and for some constant $c > 0$

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) - \frac{1}{2} c\theta(1-\theta)||\mathbf{x}-\mathbf{y}||^2 \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1$$

- A function $f: \mathcal{D} \rightarrow \Re$ is **uniformly convex** wrt function $c(\mathbf{x}) \geq 0$ (vanishing only at 0) if $\mathcal{D}$ is convex and

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) - c(\|\mathbf{x}-\mathbf{y}\|)\theta(1-\theta) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \quad ($$

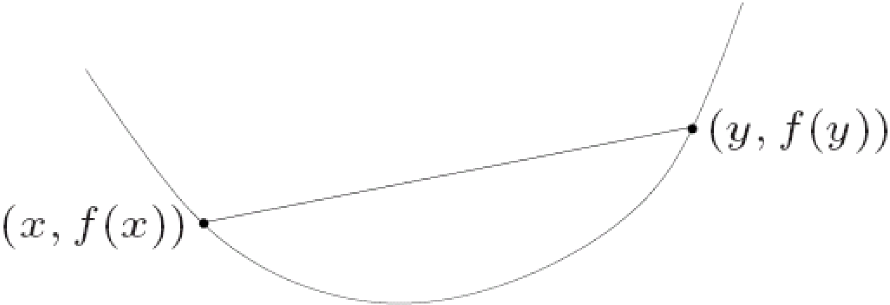


Figure 13: Example of convex function.

Examples of Convex Functions

Examples of convex functions on the set of reals $\Re$ as well as on $\Re^n$ and $\Re^{m\times n}$ are shown below.

Function type	Domain	Additional Constraints
The affine function: $ax + b$	$\Re$	Any $a, b \in \Re$
The exponential function: e^{ax}	$\Re$	Any $a \in \Re$
Powers: x^α	$\Re_{++}$	$\alpha \geq 1$ or $\alpha \leq 1$
Powers of absolute value: $\|x\|^p$	$\Re$	$p \geq 1$
Negative entropy: $x \log x$	$\Re_{++}$	
Affine functions of vectors: $\mathbf{a}^T\mathbf{x} + b$	$\Re^n$	
p-norms of vectors: $\|\|\mathbf{x}\|\|_p = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}$	$\Re^n$	$p \geq 1$
inf norms of vectors: $\|\|\mathbf{x}\|\|_\infty = \max_k \|x_k\|$	$\Re^n$	
Affine functions of matrices: $tr(A^TX) + b = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}X_{ij} + b$	$\Re^{m\times n}$	
Spectral (maximum singular value) matrix norm: $\|\|X\|\|_2 = \sigma_{max}(X) = (\lambda_{max}(X^TX))^{1/2}$	$\Re^{m\times n}$	

Table 1: Examples of convex functions on $\Re$, $\Re^n$ and $\Re^{m\times n}$.

Strict, Strong and Uniform Convexity for $f: \Re \to \Re$

- **Strictly, Strongly Convex Function:**

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \to \Re$,

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**

Strict, Strong and Uniform Convexity for $f: \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f: \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**
  - $f(x) = x^4$
  - $f(x) = x^4$
- **Convex but not Strictly Convex:**

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**
  - $f(x) = x^4$
  - $f(x) = x^4$
- **Convex but not Strictly Convex:**
  - $f(x) = |x|$

A function $f : \Re^n \to \Re$ is said to be concave if the function $-f$ is convex. Examples of concave functions on the set of reals $\Re$ are shown below. If a function is both convex and concave, it must be affine, as can be seen in the two tables.

Function type	**Domain**	**Additional Constraints**
The affine function: $ax + b$	$\Re$	Any $a, b \in \Re$
Powers: x^α	$\Re_{++}$	$0 \leq \alpha \leq 1$
logarithm: $\log x$	$\Re_{++}$	

Table 2: Examples of concave functions on $\Re$.

Convexity and Global Minimum

Fundamental chracteristics:

1. Any point of local minimum point is also a point of global minimum.
2. For any stricly convex function, the point corresponding to the gobal minimum is also unique.

To discuss these further, we need to extend the defitions of Local Minima/Maxima to arbitrary sets $\mathcal{D}$

Illustrating Local Extrema for $f : \Re^2 \rightarrow \Re$

These definitions are exactly analogous to the definitions for a function of single variable. Figure below shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.

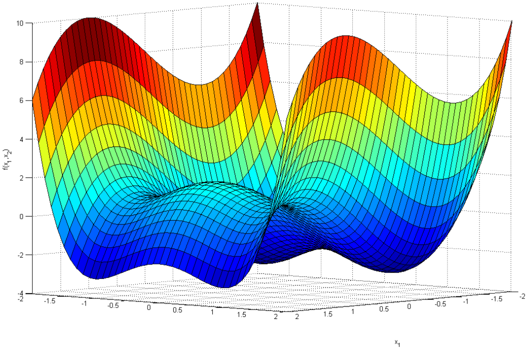


Figure 14:

Local Extrema in Normed Spaces: Extending from $\Re \rightarrow \Re$

Local Extrema in Normed Spaces: Extending from $\Re \to \Re$

Definition

[Local maximum]: *A function f of n variables has a local maximum at* $\mathbf{x}^0 \in \mathcal{D}$ *in a normed space* $\mathcal{D}$ *if* $\exists \epsilon > 0$ *such that* $\forall \ ||\mathbf{x} - \mathbf{x}^0|| < \epsilon$*.* $f(\mathbf{x}) \leq f(\mathbf{x}^0)$*. In other words,* $f(\mathbf{x}) \leq f(\mathbf{x}^0)$ *whenever* $\mathbf{x}$ *lies in the interior of some norm ball around* $\mathbf{x}^0$*.*

Definition

[Local minimum]: *A function f of n variables has a local minimum at* $\mathbf{x}^0 \in \mathcal{D}$ *in a normed space* $\mathcal{D}$ *if* $\exists \epsilon > 0$ *such that* $\forall \ ||\mathbf{x} - \mathbf{x}^0|| < \epsilon$*.* $f(\mathbf{x}) \geq f(\mathbf{x}^0)$*. In other words,* $f(\mathbf{x}) \geq f(\mathbf{x}^0)$ *whenever* $\mathbf{x}$ *lies in the interior of some norm ball around* $\mathbf{x}^0$*.*

1. These definitions can be easily extended to metric spaces or topological spaces. But we need definitions of open sets and interior in those spaces (and in fact some other foundations will also help).
2. We will first provide these defintions in $\Re^n$ and then provide the idea for extending them to more abstract topological/metric/normed spaces.

Recap: Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Balls in $\Re^n$]: Consider a point $\mathbf{x} \in \Re^n$. Then the closed norm ball around $\mathbf{x}$ of radius ϵ is

$$\mathcal{B}[\mathbf{x}, \epsilon] = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| \leq \epsilon\right\}$$

Likewise, the open nborm all around $\mathbf{x}$ of radius ϵ is defined as

$$\mathcal{B}(\mathbf{x}, \epsilon) = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| < \epsilon\right\}$$

For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when

Recap: Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Balls in $\Re^n$]: Consider a point $\mathbf{x} \in \Re^n$. Then the closed norm ball around $\mathbf{x}$ of radius ϵ is

$$\mathcal{B}[\mathbf{x}, \epsilon] = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| \leq \epsilon\right\}$$

Likewise, the open nborm all around $\mathbf{x}$ of radius ϵ is defined as

$$\mathcal{B}(\mathbf{x}, \epsilon) = \left\{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| < \epsilon\right\}$$

For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when there exists an $\epsilon > 0$ such that $\mathcal{S} \subseteq \mathcal{B}[0, \epsilon]$.

In other words, a set $\mathcal{S} \subseteq \Re^n$ is bounded means that there exists a number $\epsilon > 0$ such that for all $\mathbf{x} \in \mathcal{S}$, $||\mathbf{x}|| \leq \epsilon$.

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subseteq \mathcal{S}$.

In other words, a point $\mathbf{x} \in \mathcal{S}$ is called an interior point of a set $\mathcal{S}$ if there exists an open ball of non-zero radius around $\mathbf{x}$ such that the ball is completely contained within $\mathcal{S}$.

Definition

[Interior of a set]: Let $\mathcal{S} \subseteq \Re^n$. The set of all points

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subseteq \mathcal{S}$.

In other words, a point $\mathbf{x} \in \mathcal{S}$ is called an interior point of a set $\mathcal{S}$ if there exists an open ball of non-zero radius around $\mathbf{x}$ such that the ball is completely contained within $\mathcal{S}$.

Definition

[Interior of a set]: Let $\mathcal{S} \subseteq \Re^n$. The set of all points lying in the interior of $\mathcal{S}$ is denoted by $int(\mathcal{S})$ and is called the *interior* of $\mathcal{S}$. That is,

$$int(\mathcal{S}) = \left\{\mathbf{x} | \exists \epsilon > 0 \ s.t. \ \mathcal{B}(\mathbf{x}, \epsilon) \subset \mathcal{S}\right\}$$

In the $1-$D case, the open interval obtained by excluding endpoints from an interval $\mathcal{I}$ is the interior of $\mathcal{I}$, denoted by $int(\mathcal{I})$. For example, $int([a, b]) = (a, b)$ and $int([0, \infty)) = (0, \infty)$.

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

$$\partial(\mathcal{S}) = \left\{ \mathbf{y} | \forall\ \epsilon > 0,\ \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset \textit{ and } \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S}^C \neq \emptyset \right\}$$

For example, $partial([a, b]) = \{a, b\}$.

Definition

[Open Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is an *open set* when,

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

$$\partial(\mathcal{S}) = \left\{ \mathbf{y} | \forall\ \epsilon > 0,\ \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset \textit{ and } \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S}^C \neq \emptyset \right\}$$

For example, $partial([a, b]) = \{a, b\}$.

Definition

[Open Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is an *open set* when, for every $\mathbf{x} \in \mathcal{S}$, there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subset \mathcal{S}$.

1. The simplest examples of an open set are the open ball, the empty set $\emptyset$ and $\Re^n$.
2. Further, arbitrary union of opens sets is open. Also, finite intersection of open sets is open.
3. The interior of any set is always open. It can be proved that a set $\mathcal{S}$ is open if and only if $int(\mathcal{S}) = \mathcal{S}$.

More Basic Prerequisite Topological Concepts in $\Re^n$

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when

More Basic Prerequisite Topological Concepts in $\Re^n$

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when $\mathcal{S}^C$ (that is the complement of $\mathcal{S}$) is an open set. It can be proved that $\partial\mathcal{S} \subseteq \mathcal{S}$, that is, a closed set contains its boundary.

The closed ball, the empty set $\emptyset$ and $\Re^n$ are three simple examples of closed sets. Arbitrary intersection of closed sets is closed. Furthermore, finite union of closed sets is closed.

Definition

[Closure of a Set]: Let $\mathcal{S} \subseteq \Re^n$. The closure of $\mathcal{S}$, denoted by $closure(\mathcal{S})$ is given by

More Basic Prerequisite Topological Concepts in $\Re^n$

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when $\mathcal{S}^C$ (that is the complement of $\mathcal{S}$) is an open set. It can be proved that $\partial\mathcal{S} \subseteq \mathcal{S}$, that is, a closed set contains its boundary.

The closed ball, the empty set $\emptyset$ and $\Re^n$ are three simple examples of closed sets. Arbitrary intersection of closed sets is closed. Furthermore, finite union of closed sets is closed.

Definition

[Closure of a Set]: Let $\mathcal{S} \subseteq \Re^n$. The closure of $\mathcal{S}$, denoted by *closure*$(\mathcal{S})$ is given by

$$\textit{closure}(\mathcal{S}) = \left\{\mathbf{y} \in \Re^n | \forall\ \epsilon > 0, \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset\right\}$$

SHT: Separating hyperplane theorem (a fundamental theorem)

If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$, with a $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq \mathbf{b}$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq \mathbf{b}$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

SHT: Separating hyperplane theorem (restated)

If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$, with a $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq \mathbf{b}$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq \mathbf{b}$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

Proof of the Separating Hyperplane Theorem

We first note that the set $\mathcal{S} = \left\{\mathbf{x} - \mathbf{y} | \mathbf{x} \in \mathcal{C}, \mathbf{y} \in \mathcal{D}\right\}$ is convex, since it is the sum of two convex sets. Since $\mathcal{C}$ and $\mathcal{D}$ are disjoint, $\mathbf{0} \notin \mathcal{S}$. Consider two cases:

1. Suppose $\mathbf{0} \notin \mathit{closure}(\mathcal{S})$. Let $\mathcal{E} = \{0\}$ and $\mathcal{F} = \mathit{closure}(\mathbf{S})$. Then, the euclidean distance between $\mathcal{E}$ and $\mathcal{F}$, defined as
$\mathit{dist}(\mathcal{E};\mathcal{F}) = \mathit{inf}\left\{||\mathbf{u} - \mathbf{v}||_2 | \mathbf{u} \in \mathcal{E}, \mathbf{v} \in \mathcal{F}\right\}$
is positive, and there exists a point $\mathbf{f} \in \mathcal{F}$ that achieves the minimum distance, i.e., $||\mathbf{f}||_2 = \mathit{dist}(\mathcal{E},\mathcal{F})$. Define $\mathbf{a} = \mathbf{f}$, $\mathbf{b} = ||\mathbf{f}||_2$. Then $\mathbf{a} \neq \mathbf{0}$ and the affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} - b = \mathbf{f}^T(\mathbf{x} - \frac{1}{2}\mathbf{f})$ is nonpositive on $\mathcal{E}$ and nonnegative on $\mathcal{F}$, *i.e.*, that the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = b\right\}$ separates $\mathcal{E}$ and $\mathcal{F}$. Thus, $\mathbf{a}^T(\mathbf{x} - \mathbf{y}) > 0$ for all $\mathbf{x} - \mathbf{y} \in \mathcal{S} \subseteq \mathit{closure}(\mathcal{S})$, which implies that, $\mathbf{a}^T\mathbf{x} \geq \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and $\mathbf{y} \in \mathcal{D}$.

Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in \textit{closure}(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has empty interior, it must lie in an affine set of dimension less than *n*, and any hyperplane containing that affine set contains $\mathcal{S}$ and is a hyperplane. In other words, $\mathcal{S}$ is contained in a hyperplane $\{\mathbf{z}|\mathbf{a}^T\mathbf{z} = b\}$, which must include the origin and therefore $b = 0$. In other words, $\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and all $\mathbf{y} \in \mathcal{D}$ gives us a trivial separating hyperplane.

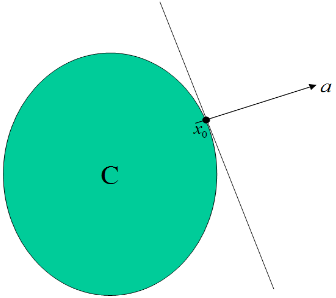
Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in closure(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has a nonempty interior, consider the set
     $\mathcal{S}_{-\epsilon} = \left\{\mathbf{z} | B(\mathbf{z}, \epsilon) \subseteq \mathcal{S}\right\}$
     where $B(\mathbf{z}, \epsilon)$ is the Euclidean ball with center $\mathbf{z}$ and radius $\epsilon > 0$. $\mathcal{S}_{-\epsilon}$ is the set $\mathcal{S}$, shrunk by ϵ. *closure* $(\mathcal{S}_{-\epsilon})$ is closed and convex, and does not contain $\mathbf{0}$, so as argued before, it is separated from $\{\mathbf{0}\}$ by atleast one hyperplane with normal vector $\mathbf{a}(\epsilon)$ such that
     $\mathbf{a}(\epsilon)^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}_{\epsilon}$
     Without loss of generality assume $||\mathbf{a}(\epsilon)||_2 = 1$. Let ϵ_k, for $k = 1, 2, \ldots$ be a sequence of positive values of ϵ_k with $\lim\limits_{k \to \infty} \epsilon_k = 0$. Since $||\mathbf{a}(\epsilon_k)||_2 = 1$ for all k, the sequence $\mathbf{a}(\epsilon_k)$ contains a convergent subsequence, and let $\overline{\mathbf{a}}$ be its limit. We have
     $\mathbf{a}(\epsilon_k)^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}_{-\epsilon_k}$
     and therefore $\overline{\mathbf{a}}^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in$ *interior*$(\mathcal{S})$, and $\overline{\mathbf{a}}^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}$, which means
     $\overline{\mathbf{a}}^T\mathbf{x} \geq \overline{\mathbf{a}}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$, and $\mathbf{y} \in \mathcal{D}$.

Supporting hyperplane theorem (consequence of separating hyperplane theorem)

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Positive Semidefinite Cone and Convex Analysis

More on Convex Sets and Advanced Material on Convex Analysis

- Positive Semi-definite cone.
- Positive Semi-definite cone: Example and Notes.
- Linear program and dual of LP.
- Properties of dual cones.
- Conic Program.
- Generalized Inequalities.

Positive semidefinite cone: Notes

1. Claim : $(S^n_+)^* = (S^n_+)$
2. i.e. $<$X,Y$>$ = tr(X^TY) = tr(XY) $\geq 0 \;\forall$ X $\in (S^n_+)$ iff Y $\in (S^n_+)$

Proof:

1.
   1. Let us say Y $\notin S^n_+$. That is $\exists$ z $\in \Re^n$ s.t. $z^T Yz =$ tr$(zz^T Y) < 0$
   2. i.e. $\exists$ X $= zz^T \in S^n_+$ s.t. $<$X,Y$> < 0$
   3. $\implies$ Y $\notin (S^n_+)^*$
2.
   1. Suppose Y,X $\in S^n_+$. Any X $\in S^n_+$ can be written in terms of eignvalue decomposition as:
   2. X $= \sum_{i=1:n} \lambda_i u_i u_i^T \; (\lambda_i \geq 0)$
   3. $\therefore <$Y,X$> =$ tr(YX) = tr(Y $\sum_{i=1:n} \lambda_i u_i u_i^T) = \sum_{i=1:n} \lambda_i tr(Y u_i u_i^T) = \sum_{i=1:n} \lambda_i u_i^T Y u_i \geq 0$.
   4. Since $(\lambda_i \geq 0)$ and $(u_i^T Y u_i \geq 0$ as Y $\in S^n_+)$
   5. $\implies$ Y $\in (S^n_+)^*$

Positive semidefinite cone: Questions

1. Q) Is there some connection between Y $= yy^T$ used for $S_+^n = \{X \in S^n \mid <yy^T, X> \geq 0\}$ and $(S_+^n)^* = (S_+^n)$.
   - (To be revisited as H/W)
2. Q) $(S_{++}^n)^* = ?$, $\text{int}(S_+^n) = (S_{++}^n)$
   - Ans: $(S_{++}^n)^* = (S_+^n)$, (will be done formally for general case of convex cones)
   - C = convex cone, $C^{**} = \text{cl(C)}$
3. Q) Consider an application of psd cone for optimization. (thru LP)
   1. We will first see (weak) duality in a linear optimization problem (LP).
   2. Next we look at generalized (conic) inequalities and the properties that the cone must satisfy for the inequality to be a valid inequality.
   3. Next, we generalize LP to conic program (CP) using generalized inequality and realize weak duality for CP thru dual cones.

Linear program (LP) & dual of LP.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in R^n_+$)
   - Then $\lambda^T(-A\mathbf{x} + b) \leq 0$
   - $\implies c^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x} + b)$
   - $\implies c^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $c^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - Thus,

$$\mathbf{c}^T\mathbf{x} \geq \begin{cases} \lambda^T\mathbf{b}, & \text{if } A^T\lambda = c \\ -\infty, & \text{otherwise} \end{cases}$$

   - Note: LHS ($\mathbf{c}^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T\mathbf{b}$) is independent of $\mathbf{x}$.

2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded) $\geq$ Dual LP (upper bounded):
   $(\min_{\mathbf{x}\in\Re^n} \mathbf{c}^T\mathbf{x}$, s.t. $A\mathbf{x} \geq \mathbf{b}) \geq (max_{\lambda\geq 0}\mathbf{b}^T\lambda$, s.t. $A^T\lambda = \mathbf{c})$

Conic program

We will motivate through linear programming (LP), generalized inequalities:

1. LP: $\min_{\mathbf{x} \in \Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$
   - Note: $-A\mathbf{x} + b \leq 0$ can be rewritten as $A\mathbf{x} \geq 0$.
   - So, constraint is $A\mathbf{x} - b \in R^n_+$
   - Note: R^n_+ is a CONE. How about defining generalized inequality for a cone K as: $c \geq_K d$ iff $c - d \in K$
2. So, a generalized conic program can be defined as:
   $\min_{\mathbf{x} \in \Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
   - That is, constraint is $A\mathbf{x} - b \in K$.

Properties of dual cones

1. If X is a Hilbert space & $C \subseteq X$ then C^* is a closed convex cone.
   - We have already proven that C^* is a closed convex cone.
   - C^* = intersection of infinite topological half spaces.
   - $C^* = \cap_{\mathbf{x} \in C} \{y | y \in X, < \mathbf{y}, \mathbf{x} > \geq 0\}$
   - $\implies C^*$ is closed.
2. $C_1 \subseteq C_2 \implies C_2^* \subseteq C_1^*$.
3. interior$(C^*) = \{\mathbf{y} \in X | < \mathbf{y}, \mathbf{x} >> 0\}$
4. If C is cone and has $int(C) \neq \emptyset$ then C^* is pointed.
   - Since; if $\mathbf{y} \in C^*$ & $-\mathbf{y} \in C^*$, then $\mathbf{y} = 0$.
5. If C is cone then closure$(C) = C^{**}$
   - If $C =$ open half space, then $C^{**} =$ closed half space.
6. If closure of C is pointed, then interior$(C^*) \neq \phi$.

S is called conically spanning set of cone K iff $conic(S) = K$.

Generalized Inequalities

a convex cone $K \subseteq \Re^n$ is a proper cone (or regular cone) if:
(Some restrictions on K that we will require, H/W Why?)

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)
  - i.e. K has no straight lines passing through O.
  - i.e. if $-a, a \in K$, then a $= 0$

examples

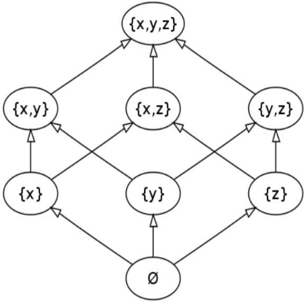
- non-negative orthant $K = R^n_+ = \{\mathbf{x} \in \Re^n | \mathbf{x}_i \geq 0, i = 1, ..., n\}$
- positive semidefinite cone $K = S^n_+$
- nonnegative polynomials on [0,1]:
  $K = \{\mathbf{x} \in \Re^n | x_1 + x_2 t + x_3 t^2 + ... + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$

Valid Inequality and Partial Order

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, reall that any partial order $\geq$ should satisfy the following properties:(refer page 51 of `www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`):

1. Reflexivity: $a \geq a$;
2. Anti-symmetry: if both $a \geq b$ and $b \geq a$, then $a = b$;
3. Transitivity: if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. Compatibility with linear operations:
    1. Homogeneity: If $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$, i.e. one can multiply both sides of an inequaility by a nonnegative real.
    2. Addititvity: if both $a \geq b$ abd $c \geq d$, then $a + c \geq b + d$, i.e. One can add two inequalities of the same sign.

Example of Partial Order

- Example of Partial Order $\subseteq$ over sets
- The Hasse diagram of the set of all subsets of a three-element set $\{\mathbf{x}, y, z\}$, ordered by inclusion(Inclusion, i.e. the Partial Order $\subseteq$):
- 

- (source `http://en.wikipedia.org/wiki/Partially_ordered_set`)

Dual Cones and Generalized Inequalities

Instructor: Prof. Ganesh Ramakrishnan

Contents: Vector Spaces beyond $\Re^n$

- Recap: Linear program (LP) & dual of LP.
- Recap: Conic program.
- Recap: Linear program (LP) & dual of LP.

Linear program (LP) & dual of LP.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x}+\mathbf{b} \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in \Re^n_+$)
   - Then $\lambda^T(-A\mathbf{x}+\mathbf{b}) \leq 0$
   - $\implies \mathbf{c}^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x}+b)$
   - $\implies \mathbf{c}^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $\mathbf{c}^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T\mathbf{b} + (\mathbf{c} - A^T\lambda)^T\mathbf{x}$
   - Thus,

$$\mathbf{c}^T\mathbf{x} \geq \begin{cases} \lambda^T b, & \text{if } A^T\lambda = \mathbf{c} \\ -\infty, & \text{otherwise} \end{cases}$$

   - Note: LHS ($c^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T b$) is independent of $\mathbf{x}$.

2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded by dual) $\geq$ Dual LP (upper bounded by primal):
   $(\min_{\mathbf{x}\in\Re^n} \mathbf{c}^T\mathbf{x}, s.t. A\mathbf{x} \geq b) \geq (max_{\lambda\geq 0} b^T\lambda, s.t. A^T\lambda = c)$

Conic program

We will motivate through linear programming (LP), generalized inequalities:

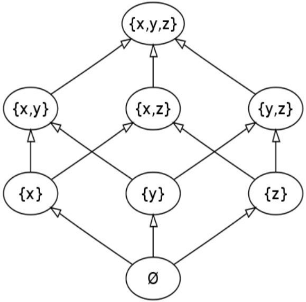
1. A generalized conic program can be defined as:
   $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x}+b \leq_K 0$
   - That is, constraint is $A\mathbf{x}-b \in K$.
2. Q: Has to generalize $-A\mathbf{x}+b \leq 0$ to $-A\mathbf{x}+b \leq_K 0$ s.t. $\leq_K$ is a generalized inequality & K some set?
3. What properties should K satisfy so that $\leq_K$ satisfies properties of generalized inequalities?

Valid Inequality and Partial Order

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, reall that any partial order $\geq$ should satisfy the following properties:(refer page 51 of `www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`):

1. Reflexivity: $a \geq a$;
2. Anti-symmetry: if both $a \geq b$ and $b \geq a$, then $a = b$;
3. Transitivity: if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. Compatibility with linear operations:
    1. Homogeneity: If $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$, i.e. one can multiply both sides of an inequaility by a nonnegative real.
    2. Addititvity: if both $a \geq b$ abd $c \geq d$, then $a + c \geq b + d$, i.e. One can add two inequalities of the same sign.

Example of Partial Order

- Example of Partial Order $\subseteq$ over sets
- The Hasse diagram of the set of all subsets of a three-element set $\{\mathbf{x}, y, z\}$, ordered by inclusion(Inclusion, i.e. the Partial Order $\subseteq$):
- 

- (source `http://en.wikipedia.org/wiki/Partially_ordered_set`)

Proof of generalized inequality

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality.

Proof:

1. K being pointed convex cone $\implies$ $\geq_K$ is a partial order
    1. Reflexivity: $a \geq_K a$, since $a - a = 0 \in K$ ($\because$ K is cone)
    2. Anti-symmetry: If a $\geq_K$ b & b $\geq_K$ a then a $=$ b, since a - b $\in$ K & b -a $\in$ K $\implies$ a - b $=$ 0 ($\because$ K is pointed)
    3. Transitivity: If both a $\geq_K$ b & b $\geq_K$ c then a $\geq_K$ c, since a - b $\in$ K & b -c $\in$ K $\implies$ (a - b) $+$ (b - c) $\in$ K ($\because$ K is a convex cone i.e. contain all conic combinations of points in the set)
    4. Homogeneity: If both a $\geq_K$ b & $\lambda \geq 0$ then λa $\geq_K$ λb, since a - b $\in$ K & $\lambda \geq 0$ $\implies$ λ(a - b) $\in$ K ($\because$ K is a cone)
    5. Additivity: If a $\geq_K$ b & c $\geq_K$ d then a $+$ c $\geq_K$ b $+$ d, since a - b $\in$ K & c -d $\in$ K $\implies$ (a $+$ c) - (b $+$ d) $\in$ K ($\because$ K is a convex cone)
2. $\geq_K$ is a partial order $\implies$ K being pointed convex cone

Proof of generalized inequality

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality.
Proof:

1. $\geq_K$ is a partial order $\implies$ K being pointed convex cone
    1. K is convex cone: If $\mathbf{x}, \mathbf{y} \in K$ then $\theta_1\mathbf{x} + \theta_2 y \in K \forall\ \theta_1, \theta_2 \geq 0$, since $\mathbf{x} \geq_K 0$ & $y \geq_K 0 \implies \theta_1\mathbf{x} \geq_K 0$ & $\theta_2 y \geq_K 0\ \forall\ \theta_1, \theta_2 \geq 0$ (Homogeneity of $\geq_K$) and thus $\theta_1\mathbf{x} + \theta_2 y \geq 0$ (Additivity of $\geq_K$)
    2. K is pointed: If $\mathbf{x} \in K$ & $-\mathbf{x} \in K$ then $\mathbf{x} = 0$, since $\mathbf{x} \geq_K \mathbf{x}$ & $-\mathbf{x} \geq_K 0 \implies 0 \geq_K \mathbf{x}$ (reflectivity $\mathbf{x} \geq_K \mathbf{x}$, and adding $\mathbf{x} \geq_K \mathbf{x} \& {-\mathbf{x}} \geq_K 0$ by additivity) and $-\mathbf{x} \geq_K \mathbf{x}$ (additivity on $-\mathbf{x} \geq_K 0$ & $0 \geq_K \mathbf{x}$) and similarly $\mathbf{x} \geq_K -\mathbf{x}$, and by applying anti-symmetry on $-\mathbf{x} \geq_K \mathbf{x}$ & $\mathbf{x} \geq_K -\mathbf{x}$ we get $\mathbf{x} = -\mathbf{x}$ i.e. $\mathbf{x} = 0$.

Additional properties over & above K being pointed convex cone

1. Que: Suppose $a^i \geq_K b^i$ $\forall$ i & $a^i \rightarrow a$ & $b^i \rightarrow b$, then for $a \geq_K b$ what more is required of K?
2. Ans: Necessary condition is that a^i - $b^i \rightarrow a - b \in$ K. i.e. K is closed(Also happens to be a sufficient condition).
3. Que: What is required so that $\exists$ $a >_K b$ (i.e. $b \ngeq_K a$)?
4. Ans: Sufficient condition is that $a - b \in$ int(K) i.e. int(K) $\neq \phi$ OR K has non-empty interior.

Linear program (LP) & Conic program.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$

$-A\mathbf{x} + b \leq 0$ can be rewritten as $A\mathbf{x} \geq b$ or $A\mathbf{x} - b \in \Re^n_+$ Note: $\Re^n_+$ is a CONE. How about defining generalized inequality for a cone C as c $>_K$ d iff $c - d \in K$ and a generl conic program as:

1. $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$

- That is, constraint is $A\mathbf{x} - b \in K$.
- K is a proper cone.

Generalized Inequalities

a convex cone $K \subseteq \Re^n$ is a proper cone (or regular cone) if:
(Some restrictions on K that we will require, H/W Why?)

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)
  - i.e. K has no straight lines passing through O.
  - i.e. if $-a, a \in K$, then a $= 0$

examples

- non-negative orthant $K = R^n_+ = \{\mathbf{x} \in \Re^n | x_i \geq 0, i = 1, ..., n\}$
- psitive semidefinite cone $K = S^n_+$
- nonnegative polynomials on [0,1]:
  $K = \{\mathbf{x} \in \Re^n | x_1 + x_2 t + x_3 t^2 + ... + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$
- Que: What if $n \rightarrow \infty$, can you get proper cones under additional constraints?

Linear program & its dual To Conic program and its dual.

Consider LP and its dual:

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x}+b \le 0$
   - Let $\lambda \ge 0$ (i.e. $\lambda \in R^n_+$)
   - Then $\lambda^T(-A\mathbf{x}+b) \le 0$
   - $\implies c^T\mathbf{x} \ge c^T\mathbf{x} + \lambda^T(-A\mathbf{x}+b)$
   - $\implies c^T\mathbf{x} \ge \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $c^T\mathbf{x} \ge \min_{\mathbf{x}} \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - Thus,

$$c^T\mathbf{x} \ge \begin{cases} \lambda^T b, & \text{if } A^T\lambda = c \\ -\infty, & \text{otherwise} \end{cases}$$

   - Note: LHS ($\mathbf{c}^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T\mathbf{b}$) is independent of $\mathbf{x}$.
2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded by dual) $\ge$ Dual LP (upper bounded by primal):
   $(\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}, s.t. A\mathbf{x} \ge b) \ge (max_{\lambda\ge 0} b^T\lambda, s.t. A^T\lambda = c)$

Conic program

Refer page 5 of `http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf`:

1. Conic program:
   $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
2. Generalized conic program:
   $\min_{\mathbf{x} \in V} < c, \mathbf{x} >_V$
   subjected to $A\mathbf{x} - b \in K$
3. K is a regular/proper cone.
4. We need an equivalent $\lambda \in D \supseteq K^*$ s.t.
   $<\lambda, A\mathbf{x} - b >\geq 0$.
5. This K^* s.t.
   $D = \{\lambda | < \lambda, A\mathbf{x} - \mathbf{b} >\geq 0, \ \lambda \in V \ \forall \ A\mathbf{x} - b \in K\}$
   & $D \supseteq K^*$ is dual cone of K

Dual of Conic program

1. Refer page 7 of `http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf`: $K^* = \{\lambda : \lambda^T \xi \geq 0 \ \forall \xi \in K\}$ is the cone dual to K.
2. With this follows weak duality theorem for CONIC PROGRAM: Primal CP (lower bounded by dual) $\geq$ Dual CP (upper bounded by primal): $(\min_{\mathbf{x} \in V} < c, \mathbf{x} >_V$, s.t. $< \lambda, A\mathbf{x} - \mathbf{b} > \geq 0.) \geq (max_{\lambda \in K^*} < b, \lambda >, s.t. A^T \lambda = c)$

Notes: LP and CP

1. Both LP and CP dealt with affine objectives.
2. CP dealt with the generalized conic inequalities.
3. Later, in convex optimization, we will deal with the more general convex functions in the objective.

Some Generalizations:

1. If $K = R_+^n$, the CP is an LP.
2. If $K = S_+^n$ (Set of all nXn SPD matrices), the CP is an SDP (Semi-definite program).
3. Any generic convex program can be expressed as a cone program (CP).

Dual of dual

1. If K is a closed convex cone then $K^{**} = K$.
2. More generally, if K is just a convex cone, $K^{**} =$ closure(K) (abbreviated as Cl(K))

We will prove that if K is closed, then $K^{**} = K$:

1. $K \subseteq K^{**}$, since $\mathbf{x} \in K \implies <\mathbf{x}, \mathbf{y}> \geq 0 \ \forall \ \mathbf{y} \in K^* \implies \mathbf{x} \in K^{**}$.
2. $K^{**} \subseteq K$, we will prove by contradiction. Suppose $\mathbf{x} \in K^{**}$ but $\mathbf{x} \notin K$:
    1. K^{**} is closed since any dual cone is intersection of half spaces that are closed.
    2. $\{\mathbf{x}\}$ is a singleton set.
    3. $\implies$ by "strict hyperplane theorem" (on next page and proved later): $\exists \mathbf{a} \in V \ \& \ \mathbf{b} \in \Re$ s.t. $<\mathbf{a}, \mathbf{x}> < \mathbf{b} \& <\mathbf{a}, \mathbf{y}> \geq b \forall \ \mathbf{y} \in K$.
    4. $\implies <\mathbf{a}, \mathbf{x}> < 0 \leq <\mathbf{a}, \mathbf{y}> \forall \mathbf{y} \in K$. (Since $\mathbf{y} = 0 \in K^{**}$, Claim: $\mathbf{b} = 0$ if V is a closed convex cone)
    5. $\implies \mathbf{a} \in K^* \ \& \ \mathbf{x} \notin K^{**}$ [contradiction]

Separating hyperplane theorem (a fundamental theorem)

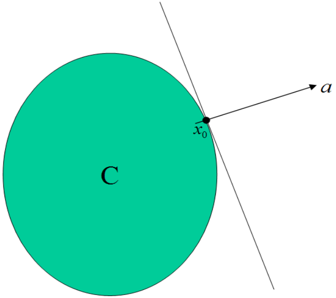
If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$, with a $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq \mathbf{b}$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq \mathbf{b}$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

Supporting hyperplane theorem (consequence of separating hyperplane theorem)

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Dual cones and generalized inequalities

In-fact, if K is a proper cone then K^* is also proper.
$K^* = \{\lambda : \lambda^T \xi \geq 0,\ \forall \xi \in K\}$ is the cone dual to K.
Examples:

- Self-dual cones
  - $K = \Re^n_+$: $K^* = \Re^n_+$
  - $K = S^n_+$: $K^* = S^n_+$
  - $K = \{(\mathbf{x}, t) | \|\mathbf{x}\|_2 \leq t\}$: $K^* = \{(\mathbf{x}, t) | \|\mathbf{x}\|_2 \leq t\}$
- $K = \{(\mathbf{x}, t) | \|\mathbf{x}\|_1 \leq t\}$: $K^* = \{(\mathbf{x}, t) | \|\mathbf{x}\|_\infty \leq t\}$

Dual cones of proper cones are proper, hence define generalized inequalities:
$\mathbf{y} \succeq_{K^*} 0 \iff \mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \succeq_K 0$

Minimum and minimal elements via dual inequalities

minimum element w.r.t $\preceq_K$:

- $\mathbf{x}$ is minimum element of S iff for all $\lambda \succ_{K^*} 0$, $\mathbf{x}$ is unique minimizer of $\lambda^T \mathbf{z}$ over S.

minimal element w.r.t $\preceq_K$:

- If $\mathbf{x}$ minimizes $\lambda^T \mathbf{z}$ over S for some $\lambda \succ_{K^*} 0$ then $\mathbf{x}$ is minimal
- If $\mathbf{x}$ is minimal element of convex set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that $\mathbf{x}$ minimizes $\lambda^T \mathbf{z}$ over S

From Dual of Norm Cone to Dual Norm

Let $\|.\|$ be a norm on $\Re^n$ The dual of $K = \{(\mathbf{x}, t) \in \Re^{n+1} | \|\mathbf{x}\| \leq t\}$ is:
$K^* = \{(u, v) | \in \Re^{n+1} | \|u\|_* \leq v\}$
where $\|u\|_* = sup\{u^T\mathbf{x} | \|\mathbf{x}\| \leq 1\}$
Proof: We need to show that
$\mathbf{x}^T u + tv \geq 0$ whenever $\|\mathbf{x}\| \leq t \iff \|u\|_* \leq v$