Definition 35 [Convex Function]: *A function $f : \mathcal{D} \rightarrow \Re$ is convex if $\mathcal{D}$ is a convex set and*

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{4.31}$$

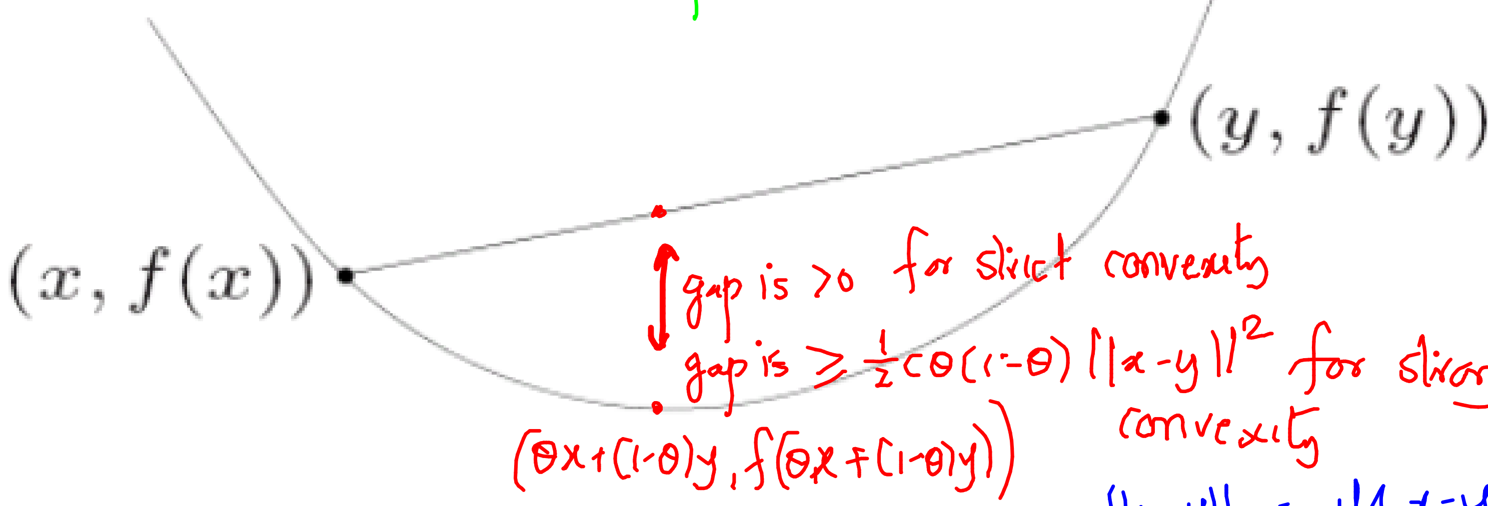
Figure 4.37 illustrates an example convex function. A function $f : \mathcal{D} \rightarrow \Re$ is strictly convex if $\mathcal{D}$ is convex and

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) < \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{4.32}$$

A function $f : \mathcal{D} \rightarrow \Re$ is called uniformly or strongly convex if $\mathcal{D}$ is convex and there exists a constant $c > 0$ such that

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) - \frac{1}{2}c\theta(1-\theta)||\mathbf{x}-\mathbf{y}||^2 \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D}$$

function at convex combination is less than convex combination of fn values & a factor dependent on distance between the pts



$||x-y|| = 0$ iff $x = y$

Figure 4.37: Example of convex function.

$\Rightarrow$ strong convexity implies strict convexity

$F(x_1, x_2, x_3) = f(x_1, x_2) - x_3$

$\nabla F = \begin{bmatrix} \nabla f \\ -1 \end{bmatrix}$

$f(x_1, x_2)$

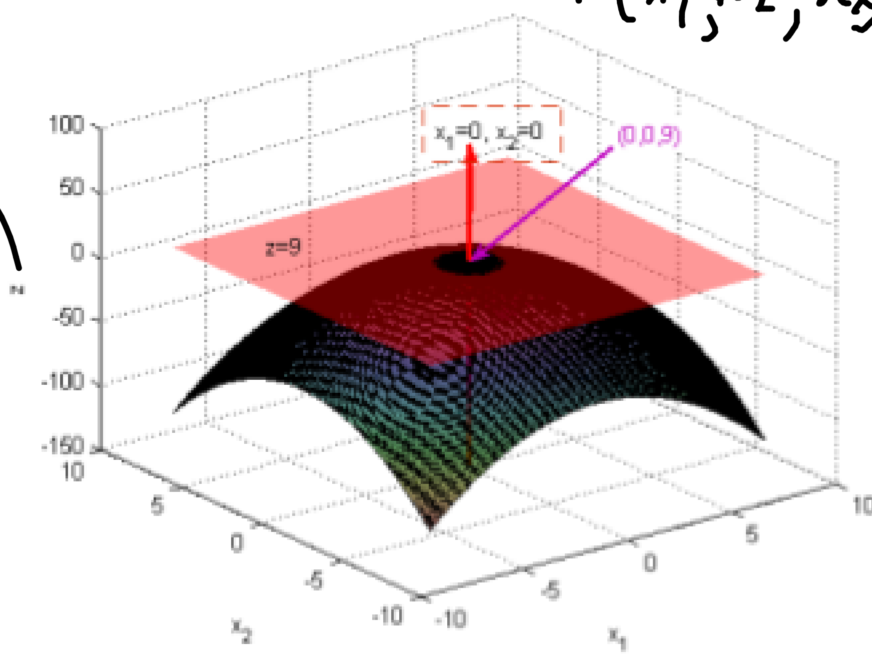


Figure 4.17: The paraboloid $f(x_1, x_2) = 9 - x_1^2 - x_2^2$ attains its maximum at $(0,0)$. The tanget plane to the surface at $(0, 0, f(0,0))$ is also shown, and so is the gradient vector ∇F at $(0, 0, f(0,0))$.

We can embed the graph of a function of n variables as the 0-level surface of a function of $n+1$ variables. More concretely, if $f : \mathcal{D} \rightarrow \Re$, $\mathcal{D} \subseteq \Re^n$ then we define $F : \mathcal{D}' \rightarrow \Re$, $\mathcal{D}' = \mathcal{D} \times \Re$ as $F(\mathbf{x}, z) = f(\mathbf{x}) - z$ with $\mathbf{x} \in \mathcal{D}'$. The function f then corresponds to a single level surface of F given by $F(\mathbf{x}, z) = 0$. In other words, the $0-$level surface of F gives back the graph of f. The gradient of F at any point $(\mathbf{x}, z)$ is simply, $\nabla F(\mathbf{x}, z) = [f_{x_1}, f_{x_2}, \ldots, f_{x_n}, -1]$ with the first n components of $\nabla F(\mathbf{x}, z)$ given by the n components of $\nabla f(\mathbf{x})$. We note that the level surface of F passing through point $(\mathbf{x}^0, f(\mathbf{x}^0)$ is its 0-level surface, which is essentially the surface of the function $f(\mathbf{x})$. The equation of the tangent hyperplane to the $0-$level surface of F at the point $(\mathbf{x}^0, f(\mathbf{x}^0)$ (that is, the tangent hyperplane to $f(\mathbf{x})$ at the point $\mathbf{x}_0$), is $\nabla F(\mathbf{x}^0, f(\mathbf{x}^0))^T.[\mathbf{x} - \mathbf{x}^0, z - f(\mathbf{x}^0)]^T = 0$. Substituting appropriate expression for $\nabla F(\mathbf{x}^0)$, the equation of the tangent plane can be written as

$$\left(\sum_{i=1}^{n} f_{x_i}(\mathbf{x}^0)(x_i - x_i^0)\right) - (z - f(\mathbf{x}^0)) = 0$$

or equivalently as,

(* -1)

$$\left(\sum_{i=1}^{n} f_{x_i}(\mathbf{x}^0)(x_i - x_i^0)\right) + f(\mathbf{x}^0) = z$$

As an example, consider the paraboloid, $f(x_1, x_2) = 9 - x_1^2 - x_2^2$, the corresponding $F(x_1, x_2, z) = 9 - x_1^2 - x_2^2 - z$ and the point $x^0 = (\mathbf{x}^0, z) = (1, 1, 7)$ which lies on the 0-level surface of F. The gradient $\nabla F(x_1, x_2, z)$ is $[-2x_1,\ -2x_2,\ -1]$, which when evaluated at $x^0 = (1, 1, 7)$ is $[-2,\ -2,\ -1]$. The equation of the tangent plane to f at x^0 is therefore given by $-2(x_1 - 1) - 2(x_2 - 1) + 7 = z$.

Theorem 60 *If $f(\mathbf{x})$ defined on a domain $\mathcal{D} \subseteq \Re^n$ has a local maximum or minimum at $\mathbf{x}^*$ and if the first-order partial derivatives exist at $\mathbf{x}^*$, then $f_{x_i}(\mathbf{x}^*) = 0$ for all $1 \le i \le n$.*

First-order condition

f is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

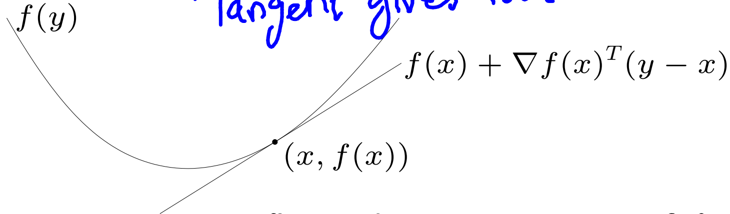
$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \mathbf{dom}\, f$

1st-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq \underbrace{f(x) + \nabla f(x)^T (y}_{} - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$

Tangent gives lower bound estimate



first-order approximation of f is global underestimator

Second-order conditions

f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

2nd-order conditions: for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex

Examples on R

convex:

- affine: $ax + b$ on **R**, for any $a, b \in$ **R** ✓
- exponential: e^{ax}, for any $a \in$ **R** $AM \geq GM$
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on **R**, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on **R**, for any $a, b \in$ **R**
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

examples on $\mathbf{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

examples on $\mathbf{R}^{m \times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

$$\max_v \frac{\|Xv\|_2}{\|v\|_2}$$

Theorem 75 *Let $f : \mathcal{D} \rightarrow \Re$ be a differentiable convex function on an open convex set $\mathcal{D}$. Then:*

1. *f is convex if and only if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \tag{4.44}$$

2. *f is strictly convex on $\mathcal{D}$ if and only if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, with $\mathbf{x} \neq \mathbf{y}$,*

$$f(\mathbf{y}) > f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \tag{4.45}$$

3. *f is strongly convex on $\mathcal{D}$ if and only if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \frac{1}{2}c||\mathbf{y} - \mathbf{x}||^2 \tag{4.46}$$

   for some constant $c > 0$.

Proof:

Sufficiency: The proof of sufficiency is very similar for all the three statements of the theorem. So we will prove only for statement (4.44). Suppose (4.44) holds. Consider $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$ and any $\theta \in (0,1)$. Let $\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2$. Then,

$$\begin{aligned} f(\mathbf{x}_1) &\geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{x}_1 - \mathbf{x}) \\ f(\mathbf{x}_2) &\geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{x}_2 - \mathbf{x}) \end{aligned} \tag{4.47}$$

Adding $(1-\theta)$ times the second inequality to θ times the first, we get,

$$\theta f(\mathbf{x}_1) + (1-\theta) f(\mathbf{x}_2) \geq f(\mathbf{x})$$

which proves that $f(\mathbf{x})$ is a convex function. In the case of strict convexity, strict inequality holds in (4.47) and it follows through. In the case of strong convexity, we need to additionally prove that

$$\theta \frac{1}{2} c||\mathbf{x} - \mathbf{x}_1||^2 + (1-\theta)\frac{1}{2}c||\mathbf{x} - \mathbf{x}_2||^2 = \frac{1}{2}c\theta(1-\theta)||\mathbf{x}_2 - \mathbf{x}_1||^2$$

Necessity: Suppose f is convex. Then for all $\theta \in (0,1)$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, we must have

$$f(\theta\mathbf{x}_2 + (1-\theta)\mathbf{x}_1) \leq \theta f(\mathbf{x}_2) + (1-\theta) f(\mathbf{x}_1)$$

Thus,

$$\nabla^T f(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1) = \lim_{\theta \to 0} \frac{f(\mathbf{x}_1 + \theta(\mathbf{x}_2 - \mathbf{x}_1)) - f(\mathbf{x}_1)}{\theta} \leq f(\mathbf{x}_2) - f(\mathbf{x}_1)$$

This proves necessity for (4.44). The necessity proofs for (4.45) and (4.46) are very similar, except for a small difference for the case of strict convexity; the strict inequality is not preserved when we take limits. Suppose equality does hold in the case of strict convexity, that is for a strictly convex function f, let

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla^T f(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1) \tag{4.48}$$

for some $\mathbf{x}_2 \neq \mathbf{x}_1$. Because f is stricly convex, for any $\theta \in (0,1)$ we can write

$$f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) = f(\mathbf{x}_2 + \theta(\mathbf{x}_1 - \mathbf{x}_2)) < \theta f(\mathbf{x}_1) + (1-\theta) f(\mathbf{x}_2) \tag{4.49}$$

Since (4.44) is already proved for convex functions, we use it in conjunction with (4.48), and (4.49), to get

$$f(\mathbf{x}_2) + \theta\nabla^T f(\mathbf{x}_2)(\mathbf{x}_1 - \mathbf{x}_2) \leq f(\mathbf{x}_2 + \theta(\mathbf{x}_1 - \mathbf{x}_2)) < f(\mathbf{x}_2) + \theta\nabla^T f(\mathbf{x}_2)(\mathbf{x}_1 - \mathbf{x}_2)$$

which is a contradiction. Thus, equality can never hold in (4.44) for any $\mathbf{x}_1 \neq \mathbf{x}_2$. This proves the necessity of (4.45). $\square$

Definition 41 [Subgradient]: *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A vector $\mathbf{h} \in \Re^n$ is said to be a subgradient of f at the point $\mathbf{x} \in \mathcal{D}$ if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{y} \in \mathcal{D}$. The set of all such vectors is called the subdifferential of f at $\mathbf{x}$.

Theorem 76 *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A point $\mathbf{x} \in \mathcal{D}$ corresponds to a minimum if and only if*

$$\nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \geq 0$$

for all $\mathbf{y} \in \mathcal{D}$.

If $\nabla f(\mathbf{x})$ is nonzero, it defines a supporting hyperplane to $\mathcal{D}$ at the point $\mathbf{x}$. Theorem 77 implies that for a differentiable convex function defined on an open set, every critical point must be a point of (global) minimum.

Theorem 77 *Let $f : \mathcal{D} \rightarrow \Re$ be differentiable and convex on an open convex domain $\mathcal{D} \subseteq \Re^n$. Then $\mathbf{x}$ is a critical point of f if and only if it is a (global) minimum.*

Theorem 78 *Let $f : \mathcal{D} \rightarrow \Re$ with $\mathcal{D} \subseteq \Re^n$ be differentiable on the convex set $\mathcal{D}$. Then,*

1. *f is convex on $\mathcal{D}$ if and only if is its gradient ∇f is monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$*

$$\left(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\right)^T (\mathbf{x} - \mathbf{y}) \geq 0 \tag{4.53}$$

2. *f is strictly convex on $\mathcal{D}$ if and only if is its gradient ∇f is strictly monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$ with $\mathbf{x} \neq \mathbf{y}$,*

$$\left(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\right)^T (\mathbf{x} - \mathbf{y}) > 0 \tag{4.54}$$

3. *f is uniformly or strongly convex on $\mathcal{D}$ if and only if is its gradient ∇f is uniformly monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$,*

$$\left(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\right)^T (\mathbf{x} - \mathbf{y}) \geq c||\mathbf{x} - \mathbf{y}||^2 \tag{4.55}$$

for some constant $c > 0$.

Necessity: Suppose f is uniformly convex on $\mathcal{D}$. Then from theorem 75, we know that for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) - \frac{1}{2}c||\mathbf{y} + \mathbf{x}||^2$$
$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla^T f(\mathbf{y})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}c||\mathbf{x} + \mathbf{y}||^2$$

Adding the two inequalities, we get (4.55). If f is convex, the inequalities hold with $c = 0$, yielding (4.54). If f is strictly convex, the inequalities will be strict, yielding (4.54).

Sufficiency: Suppose ∇f is monotone. For any fixed $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, consider the function $\phi(t) = f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$. By the mean value theorem applied to $\phi(t)$, we should have for some $t \in (0, 1)$,

$$\phi(1) - \phi(0) = \phi'(t) \tag{4.56}$$

Letting $\mathbf{z} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$, (4.56) translates to

$$f(\mathbf{y}) - f(\mathbf{x}) = \nabla^T f(\mathbf{z})(\mathbf{y} - \mathbf{x}) \tag{4.57}$$

Also, by definition of monotonicity of ∇f, (from (4.53)),

$$\left(\nabla f(\mathbf{z}) - \nabla f(\mathbf{x})\right)^T (\mathbf{y} - \mathbf{x}) = \frac{1}{t}\left(\nabla f(\mathbf{z}) - \nabla f(\mathbf{x})\right)^T (\mathbf{z} - \mathbf{x}) \geq 0 \tag{4.58}$$

Combining (4.57) with (4.58), we get,

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &= \left(\nabla f(\mathbf{z}) - f(\mathbf{x})\right)^T (\mathbf{y} - \mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \\ &\geq \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \end{aligned} \tag{4.59}$$

By theorem 75, this inequality proves that f is convex. Strict convexity can be similarly proved by using the strict inequality in (4.58) inherited from strict monotonicity, and letting the strict inequality follow through to (4.59). For the case of strong convexity, from (4.55), we have

$$\begin{aligned} \phi'(t) - \phi'(0) &= \left(\nabla f(\mathbf{z}) - f(\mathbf{x})\right)^T (\mathbf{y} - \mathbf{x}) \\ = \frac{1}{t}\left(\nabla f(\mathbf{z}) - f(\mathbf{x})\right)^T (\mathbf{z} - \mathbf{x}) &\geq \frac{1}{t}c||\mathbf{z} - \mathbf{x}||^2 = ct||\mathbf{y} - \mathbf{x}||^2 \end{aligned} \tag{4.60}$$

$$\phi(1) - \phi(0) - \phi'(0) = \int_0^1 [\phi'(t) - \phi'(0)]dt \geq \frac{1}{2}c||\mathbf{y} - \mathbf{x}||^2 \tag{4.61}$$

which translates to

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \frac{1}{2}c||\mathbf{y} - \mathbf{x}||^2$$

Basic inequality

recall basic inequality for convex differentiable f:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

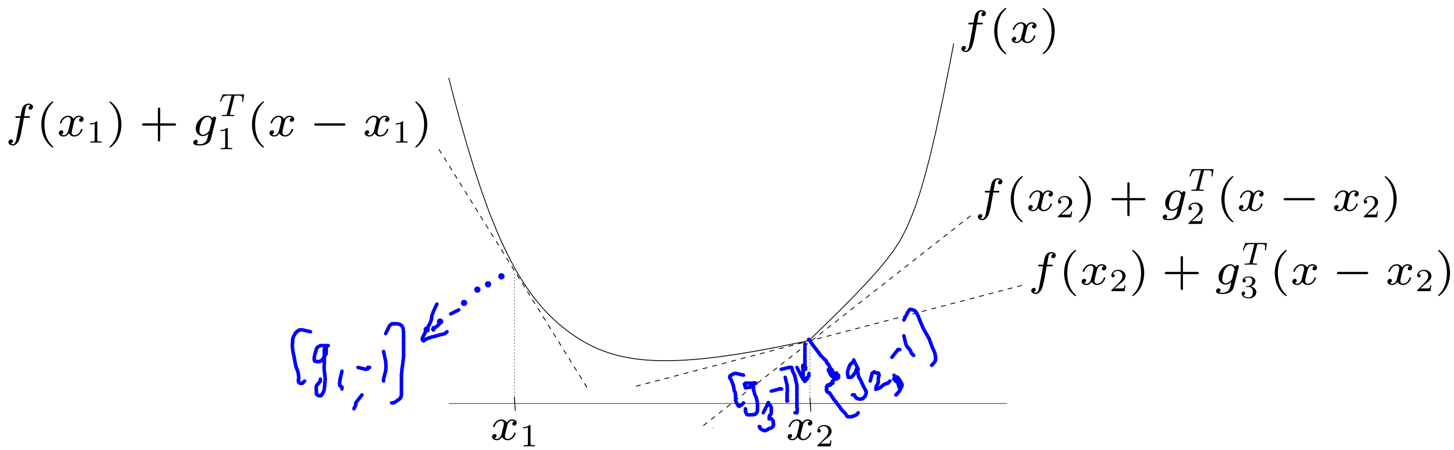
- first-order approximation of f at x is global underestimator
- $(\nabla f(x), -1)$ supports $\mathbf{epi}\, f$ at $(x, f(x))$

what if f is not differentiable?

Subgradient of a function

g is a **subgradient** of f (not necessarily convex) at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



g_2, g_3 are subgradients at x_2; g_1 is a subgradient at x_1

equivalent defns motivated by $\nabla f(x)$

- g is a subgradient of f at x iff $(g, -1)$ supports $\mathbf{epi}\, f$ at $(x, f(x))$
- g is a subgradient iff $f(x) + g^T(y - x)$ is a global (affine) underestimator of f
- if f is convex and differentiable, $\nabla f(x)$ is a subgradient of f at x

subgradients come up in several contexts:

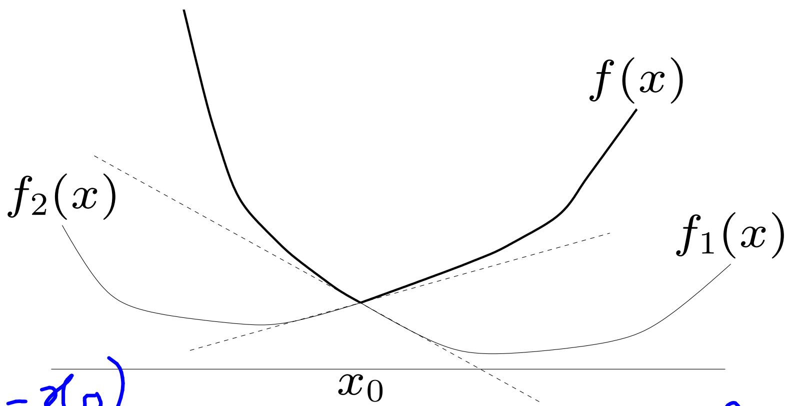
- algorithms for nondifferentiable convex optimization
- convex analysis, $e.g.$, optimality conditions, duality for nondifferentiable problems

(if $f(y) \leq f(x) + g^T(y - x)$ for all y, then g is a **supergradient**)

Example

$f = \max\{f_1, f_2\}$, with f_1, f_2 convex and differentiable

easy to see convexity



$f(y) \geq f_1(y)$

$$f(y) \geq \underbrace{f(x_0)}_{f_1(x_0)} + \nabla f_1(x_0)^T (y - x_0) \quad \forall y$$

At x_0 $f(x_0) = f_1(x_0) = f_2(x_0)$

- $f_1(x_0) > f_2(x_0)$: unique subgradient $g = \nabla f_1(x_0)$
- $f_2(x_0) > f_1(x_0)$: unique subgradient $g = \nabla f_2(x_0)$ — similar
- $f_1(x_0) = f_2(x_0)$: subgradients form a line segment $[\nabla f_1(x_0), \nabla f_2(x_0)]$

$$f(y) \geq \underbrace{f(x_0)}_{\theta f_1(x_0) + (1-\theta) f_2(x_0)} + [\theta \nabla f_1(x_0) + (1-\theta) \nabla f_2(x_0)]^T (y - x_0) \quad \forall \theta \in [0,1]$$

$$f(y) = \theta f(y) + (1-\theta) f(y) \geq \theta f_1(y) + (1-\theta) f_2(y) \geq \theta [f_1(x_0) + \nabla^T f_1(x_0)(y - x_0)] + (1-\theta)[f_2(x_0) + \nabla^T f_2(x_0)(y - x_0)]$$