Convex Sets

Convex sets

- Revisiting Affine Sets
- **Primal (V)** and **Dual (H)** Description
- Operations that preserve convexity
- Generalized inequalities
- Separating and supporting hyperplanes
- Dual cones and generalized inequalities

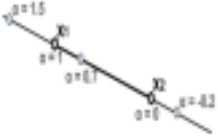
Affine combination, Affine hull and Dimension: **Primal (V) Description**

- **Affine Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k$$

$$\text{with} \quad \sum_i \theta_i = 1$$

- **Affine hull or aff(S):** The set that contains all affine combinations of points in set S = Smallest affine set that contains S.



-

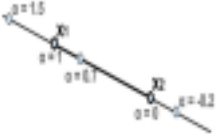
Affine combination, Affine hull and Dimension: **Primal (V) Description**

- **Affine Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k$$

$$\text{with} \quad \sum_i \theta_i = 1$$

- **Affine hull or aff(S):** The set that contains all affine combinations of points in set S = Smallest affine set that contains S.



-
- Dimension of a set S = dimension of $aff(S)$ = dimension of vector space V such that $aff(S) = \mathbf{v} + V$ for some $\mathbf{v} \in aff(S)$.
- $S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is set of $n + 1$ *affinely independent* points if $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0\}$ are linearly independent.

Recap: **Dual Representation**

If

- vector subspace $S \subseteq V$ and
- $< . >$ is an inner product on V and
- $S^{\perp}$ is orthogonal complement of S and
- $\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_K\}$ is finite spanning set in $S^{\perp}$

wrt inner product

Then:-

- $S = (S^{\perp})^{\perp} = \{\mathbf{x} | \mathbf{q}_i^T \mathbf{x} = 0; i = 1, ..., K\}$, where $K = dim(S)$
- A dual representation of vector subspace S ($\subseteq \Re^n$): $\{\mathbf{x} | Q\mathbf{x} = 0;\ \mathbf{q}_i^T$ is the i^{th} row of $Q\}$
- What about dual representations for Affine Sets, Convex Sets, Convex Cones, etc?

Recap: **Dual Representations** of Affine Sets

Recall affine set (say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector subspace $S \subseteq \Re^n$, A is affine iff:
  $A(= S \text{ shifted by } \mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n \text{ is fixed and } \mathbf{v} \in S\ \}$.
- Procedure: Let $\mathbf{u}$ be some element in the affine set A. Then $S(= A \text{ shifted by } -\mathbf{u}) = \{\ \mathbf{v} - \mathbf{u} | \mathbf{v} \in A\ \}$ is a vector subspace which has a **dual representation** $\{\mathbf{x} | Q\mathbf{x} = 0\}$
- The **dual representation for** A is therefore

$$\{x \mid Qx = Qu\}$$

Recap: **Dual Representations** of Affine Sets

Recall affine set (say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector subspace $S \subseteq \Re^n$, A is affine iff:
  $A(= S$ shifted by $\mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n$ is fixed and $\mathbf{v} \in S\ \}$.
- Procedure: Let $\mathbf{u}$ be some element in the affine set A. Then $S(= A$ shifted by $-\mathbf{u}) = \{\ \mathbf{v} - \mathbf{u} | \mathbf{v} \in A\ \}$ is a vector subspace which has a **dual representation** $\{\mathrm{x} | Q\mathrm{x} = 0\}$
- The **dual representation for** A is therefore $\{\mathrm{x} | Q\mathrm{x} = Q\mathrm{u}\}$

Recap: **Dual Representations** of Affine Sets

- For some Q with $rank = n - dim(V)$ and $\mathbf{u}$, A is affine iff: $A = \{\mathbf{x} | Q\mathbf{x} = Q\mathbf{u}\}$ i.e. solution set of linear equations represented by $Q\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = Q\mathbf{u}$.
- Example: In 3-d if Q has rank 1, we will get either a plane as solution or no solution. If Q has rank 2, we can get a plane, a line or no solution.
- Thus hyperplanes are affine spaces of dimension $n - 1$ with $Q\mathbf{x} = \mathbf{b}$ given by $p^T\mathbf{x} = \mathbf{b}$.

What is a primal (V) representation for hyperplane?

Hyperplane: **Primal (V)** and **Dual (H)** Descriptions

- **Primal (V)** Description: A hyperplane is an affine set whose dimension is one less than that of the space to which belongs. If a space is 3-dimensional then its hyperplanes are the 2-dimensional planes, while if the space is 2-dimensional, its hyperplanes are the 1-dimensional lines.
- **Dual (H)** Description: Affine set of the form $\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\}$ $(\mathbf{a} \neq 0)$

a in R^n

  - where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$
  - Alternatively: $\{\mathbf{x}|(\mathbf{x} - \mathbf{x}_0) \perp \mathbf{a}\}$, where $\mathbf{a}$ is normal and $\mathbf{x}_0 \in H$

Recap: Convex set

- **In 2D, a line segment** between distinct points $\mathbf{x}_1$,$\mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.

$$\begin{aligned} x = &\quad \alpha x_1 + \beta x_2 \\ \text{where} &\quad \alpha + \beta = 1, 0 \leq \alpha \leq 1 (\mathit{also}, 0 \leq \beta \leq 1). \end{aligned}$$

- **Convex set** : $\mathbf{x}_1, \mathbf{x}_2 \in C, 0 \leq \alpha \leq 1 \Rightarrow \alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in C$

need
direct
line seg
connections

  - Convex set is connected. Convex set can but not necessarily contains 'O'

Is every affine set convex? Is the reverse true?

Convex Combinations and Convex Hull: **Primal (V) Description**

- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- **Convex hull or conv(S)** is the set of all convex combinations of point in the set S.

→ originally non-convex

- Should S be always convex? NO
- What about the convexity of conv(S)? YES

Convex Combinations and Convex Hull: **Primal (V) Description**

- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2 + ... + \theta_k \mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- **Convex hull or conv(S)** is the set of all convex combinations of point in the set S.

- Should S be always convex? No.
- What about the convexity of conv(S)? It's always convex.

Convex Combinations and Convex Hull: **Primal (V) Description**

- Equivalent Definition of Convex Set: C is convex iff it is closed under generalized convex combinations.
- $conv(S)$ = The smallest convex set that contains S. S may not be convex but $conv(S)$ is.
  - Suppose a point lies in another smallest convex set, and not in $conv(S)$. Show that it must lie in $conv(S)$, leading to a contradiction. Proof by contradiction

- The idea of convex combination can be generalized to include infinite sums, integrals, and, in most general form, probability distributions.

as in case of prob den functions

Eg: E[X^2] where X is Gaussian distributed R.V

HW Illustrated: **Primal** and **Dual** Descriptions for Convex Polytope P

Primal or V Description : P is convex hull of finite # of points. Formally, if $\exists\ S \subset P$ s.t. $|S|$ is finite and $P = conv(S)$, then P is a **Convex Polytope**.

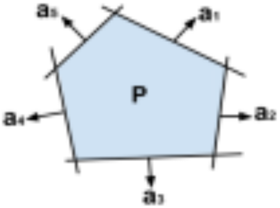
HW Illustrated: **Primal** and **Dual** Descriptions for Convex Polytope P

Primal or V Description : P is convex hull of finite # of points. Formally, if $\exists\ S \subset P$ s.t. $|S|$ is finite and $P = conv(S)$, then P is a **Convex Polytope**.
Convex hull of $n+1$ affinely independent points $\Rightarrow$ **Simplex**. It is the

generalization to $\Re^n$ of the

HW Illustrated: **Primal** and **Dual** Descriptions for Convex Polytope P

Primal or V Description : P is convex hull of finite # of points. Formally, if $\exists\ S \subset P$ s.t. $|S|$ is finite and $P = conv(S)$, then P is a **Convex Polytope**.
Convex hull of $n+1$ affinely independent points $\Rightarrow$ **Simplex**. It is the generalization to $\Re^n$ of the triangle



Dual or H Description: P is solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$ such that P is also **bounded** (else we may get half space which is not a polytope)

- $A \in \Re^{m\times n}$, $C \in \Re^{p\times n}$, $\preceq$ is component wise inequality

P is an intersection of finite number of half-spaces and hyperplanes.

Equality is for projecting a potentially higher dimensional polytope onto lower dimensional plane

Boundedness in $\Re^n$

Definition

[Balls in $\Re^n$]: Consider a point $\mathbf{x} \in \Re^n$. Then the closed ball around $\mathbf{x}$ of radius ϵ is

$$\mathcal{B}[\mathbf{x}, \epsilon] = \{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| \leq \epsilon\}$$

Likewise, the open ball around $\mathbf{x}$ of radius ϵ is defined as

$$\mathcal{B}(\mathbf{x}, \epsilon) = \{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| < \epsilon\}$$

For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when there exists an $\epsilon > 0$ such that $\mathcal{S} \subseteq \mathcal{B}[0, \epsilon]$.

In other words, a set $\mathcal{S} \subseteq \Re^n$ is bounded means that there exists an $\epsilon > 0$ such that for all $\mathbf{x} \in \mathcal{S}$, $||\mathbf{x}|| \leq \epsilon$.

Simplex (plural: simplexes) Polytope: **Primal** and **Dual** Descriptions

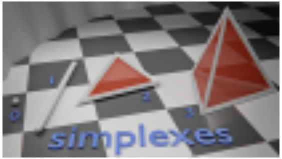


Figure 10: Source:Wikipedia

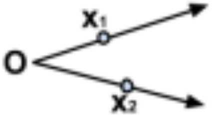
Dual or *H* Description: An n Simplex S is a convex Polytope of affine dimension n and having $n+1$ corners.

Primal or *V* Description: Convex hull of $n+1$ affinely independent points. Specifically, let $S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$ be a set of $n+1$ affinely independent points, then an n-dimensional simplex is $conv(S)$.

Simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

IS THERE ANOTHER NOTION OF HULL THAT CAN HELP US CONSTRUCTED UNBOUNDED SETS SUCH AS HALF SPACES?

Cone, conic combination and convex cone



- **Cone** A set C is a cone if $\forall \mathbf{x} \in C$, $\alpha \mathbf{x} \in C$ for $\alpha \geq 0$.
- **Conic (nonnegative) combination** of points $\mathbf{x}_1, \mathbf{x}_2$ is any point $\mathbf{x}$ of the form

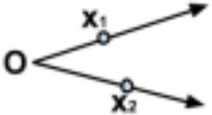
$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\text{with} \quad \alpha, \beta \geq 0.$$

Example : Diagonal vector of a parallelogram is a conic combination of the two vectors (points) $\mathbf{x}_1$ and $\mathbf{x}_2$ forming the sides of the parallelogram.

Are cones closed under conic combinations?
NO

Cone, conic combination and convex cone



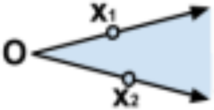
- **Cone** A set C is a cone if $\forall \mathbf{x} \in C$, $\alpha \mathbf{x} \in C$ for $\alpha \geq 0$.
- **Conic (nonnegative) combination** of points $\mathbf{x}_1, \mathbf{x}_2$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\text{with} \quad \alpha, \beta \geq 0.$$

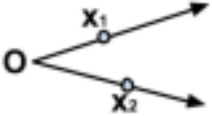
  Example : Diagonal vector of a parallelogram is a conic combination of the two vectors (points) $\mathbf{x}_1$ and $\mathbf{x}_2$ forming the sides of the parallelogram.

- **Convex cone:** The set that contains all conic combinations of points in the set.



For example, a half-space can be obtained as the set of all conic combinations of n affinely independent points and a point p lying strictly inside the half space

Conic combinations and Conic Hull



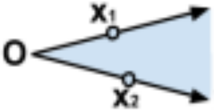
- Recap **Cone**: A set C is a cone if $\forall \mathbf{x} \in C$, $\theta \mathbf{x} \in C$ for $\theta \geq 0$.
- **Conic (nonnegative) Combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2 + ... + \theta_k \mathbf{x}_k$$

$$\text{with} \quad \theta_i \geq 0.$$

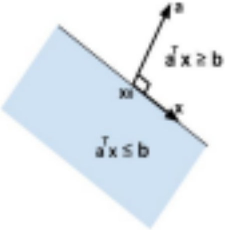
- **Conic hull or** ***conic*****(S):** The set that contains all conic combinations of points in set S.

For example, a half-space can be obtained as
affine transformation of a
conic hull of n affinely independent points
and a point p lying strictly inside the half space

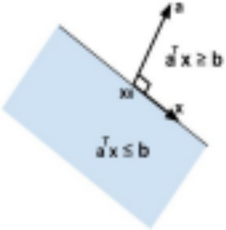


- $conic(S)$ = Smallest conic set that contains S.

From Hyperplane to Halfspace



From Hyperplane to Halfspace

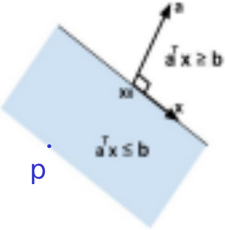


Halfspaces with b=0 are convex cones..

Dual or *H* Description: Convex cone of the form $\{\mathbf{x}|\mathbf{a}^T\mathbf{x} \leq \mathbf{b}\}$ $(\mathbf{a} \neq 0)$

- where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$

From Hyperplane to Halfspace



Dual or *H* Description: Convex cone of the form $\{\mathbf{x}|\mathbf{a}^T\mathbf{x} \leq \mathbf{b}\}$ $(\mathbf{a} \neq 0)$

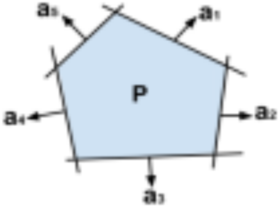
- where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$

Primal or *V* Description: Affine transformation of **conic hull** of points x and x_0 on the hyperplane **and of point p lying strictly inside the half-space**.

Convex Polyhedron

- Solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$
  - $A \in \Re^{m \times n}$
  - $C \in \Re^{p \times n}$
  - $\preceq$ is component wise inequality

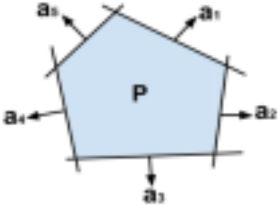
BUT THE SET NEED NOT BE BOUNDED



- This is a **Dual or H Description**: Intersection of finite number of half-spaces and hyperplanes.
- **Primal or V Description**: Can you define convex polyhedra in terms of hulls?
  1. Convex hull of finite # of points ⇒ **Convex Polytope**
  2. Convex hull of $n+1$ affinely independent points ⇒ **Simplex**
  3. Conic hull of finite # of points ⇒ POLYHEDRAL CONE

Convex Polyhedron

- Solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$
  - $A \in \Re^{m \times n}$
  - $C \in \Re^{p \times n}$
  - $\preceq$ is component wise inequality



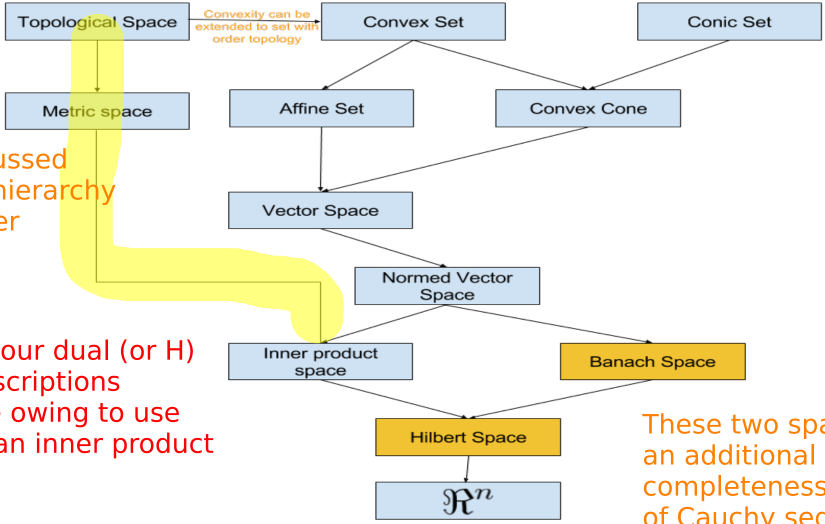
- This is a **Dual or *H* Description**: Intersection of finite number of half-spaces and hyperplanes.
- **Primal or *V* Description**: Can you define convex polyhedra in terms of hulls?
  1. Convex hull of finite # of points ⇒ **Convex Polytope**
  2. Convex hull of $n + 1$ affinely independent points ⇒ **Simplex**
  3. Conic hull of finite # of points ⇒ **Polyhedral Cone**

Polyhedral Cone: **Primal** and **Dual** Descriptions

Dual or H Description : A Polyhedral Cone P is a Convex Polyhedron with $\mathbf{b} = 0$. That is, $\{\mathbf{x} | A\mathbf{x} \succeq 0\}$ where $A \in \Re^{m \times n}$ and $\succeq$ is component wise inequality.

Primal of V Description : If $\exists$ $S \subset$ P s.t. $|S|$ is finite and $P = cone(S)$, then P is a **Polyhedral Cone**.

Homework: Structure of Mathematical Spaces Discussed (arrow means 'instance')



Here is where 0 starts belonging to the set mandatorily

Discussed this hierarchy earlier

All our dual (or H) descriptions are owing to use of an inner product

These two spaces need an additional concept of completeness (convergence of Cauchy sequences)

More Convex Sets (illustrated in $\Re^n$)

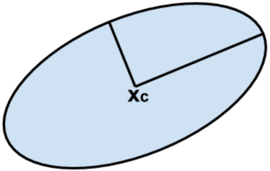
More Convex Sets (illustrated in $\Re^n$)

- Euclidean balls and ellipsoids.
- Norm balls and norm cones.
- Compact representation of vector space.
- Dual Representation.
- Different Representations of Affine Sets

Euclidean balls and ellipsoids

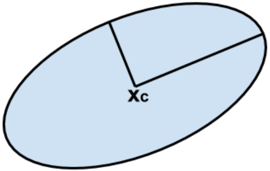
- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \leq 1 \}$
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1 \}$, where P $\in S^n_{++}$ i.e. P is positive-definite matrix.
  - Other representation: $\{\mathbf{x}_c + A\mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\}$ with A square and non-singular (*i.e.*, A^{-1} exists).

These are primal representations
What about their dual representations?



Euclidean balls and ellipsoids

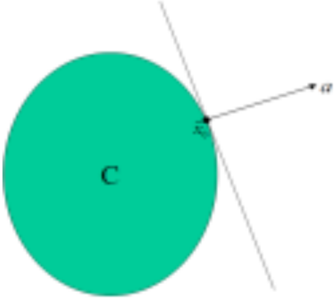
- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \le r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \le 1\ \}$
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \le 1\ \}$, where P $\in S^n_{++}$ i.e. P is positive-definite matrix.
  - Other representation: $\{\mathbf{x}_c + A\mathbf{u} \mid \|\mathbf{u}\|_2 \le 1\}$ with A square and non-singular (*i.e.*, A^{-1} exists).



Supporting hyperplane theorem and **Dual (H) Description**

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Convex set could be thought of as intersection of a possibly infinite number of half spaces created by such supporting hyperplanes

Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Homework: Separating and Supporting Hyperplane Theorems (Fill in the Blanks)

SHT: Separating hyperplane theorem (a fundamental theorem)

If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$ and $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq b$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq b$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = b\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

Proof of the Separating Hyperplane Theorem

We first note that the set $\mathcal{S} = \left\{\mathbf{x} - \mathbf{y} | \mathbf{x} \in \mathcal{C}, \mathbf{y} \in \mathcal{D}\right\}$ is convex, since it is the sum of two convex sets. Since $\mathcal{C}$ and $\mathcal{D}$ are disjoint, $\mathbf{0} \notin \mathcal{S}$. Consider two cases:

1. Suppose $\mathbf{0} \notin \mathit{closure}(\mathcal{S})$. Let $\mathcal{E} = \{0\}$ and $\mathcal{F} = \mathit{closure}(\mathbf{S})$. Then, the euclidean distance between $\mathcal{E}$ and $\mathcal{F}$, defined as
$\mathit{dist}(\mathcal{E}; \mathcal{F}) = \mathit{inf}\left\{||\mathbf{u} - \mathbf{v}||_2 | \mathbf{u} \in \mathcal{E}, \mathbf{v} \in \mathcal{F}\right\}$
is positive, and there exists a point $\mathbf{f} \in \mathcal{F}$ that achieves the minimum distance, i.e., $||\mathbf{f}||_2 = \mathit{dist}(\mathcal{E}, \mathcal{F})$. Define ______________________________.
Then $\mathbf{a} \neq \mathbf{0}$ and the affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} - b = \mathbf{f}^T(\mathbf{x} - \frac{1}{2}\mathbf{f})$ is nonpositive on $\mathcal{E}$ and nonnegative on $\mathcal{F}$, *i.e.*, that the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = b\right\}$ separates $\mathcal{E}$ and $\mathcal{F}$. Thus, $\mathbf{a}^T(\mathbf{x} - \mathbf{y}) > 0$ for all $\mathbf{x} - \mathbf{y} \in \mathcal{S} \subseteq \mathit{closure}(\mathcal{S})$, which implies that, $\mathbf{a}^T\mathbf{x} \geq \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and $\mathbf{y} \in \mathcal{D}$.

Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in \textit{closure}(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has empty interior, it must lie in an affine set of dimension less than n, and any hyperplane containing that affine set contains $\mathcal{S}$ and is a hyperplane. In other words, $\mathcal{S}$ is contained in a hyperplane $\{\mathbf{z} | \mathbf{a}^T\mathbf{z} = b\}$, which must include the origin and therefore $b = 0$. In other words, $\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and all $\mathbf{y} \in \mathcal{D}$ gives us a trivial separating hyperplane.

Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in$ *closure*$(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has a nonempty interior, consider the set
     $\mathcal{S}_{-\epsilon} = \{\mathbf{z} | B(\mathbf{z}, \epsilon) \subseteq \mathcal{S}\}$
     where $B(\mathbf{z}, \epsilon)$ is the Euclidean ball with center $\mathbf{z}$ and radius $\epsilon > 0$. $\mathcal{S}_{-\epsilon}$ is the set $\mathcal{S}$, shrunk by ϵ. *closure*$(\mathcal{S}_{-\epsilon})$ is closed and convex, and does not contain $\mathbf{0}$, so as argued before, it is separated from $\{\mathbf{0}\}$ by atleast one hyperplane with normal vector $\mathbf{a}(\epsilon)$ such that

---

Without loss of generality assume $||\mathbf{a}(\epsilon)||_2 = 1$. Let ϵ_k, for $k = 1, 2, \ldots$ be a sequence of positive values of ϵ_k with $\lim_{k \to \infty} \epsilon_k = 0$. Since $||\mathbf{a}(\epsilon_k)||_2 = 1$ for all k, the sequence $\mathbf{a}(\epsilon_k)$ contains a convergent subsequence, and let $\bar{\mathbf{a}}$ be its limit. We have

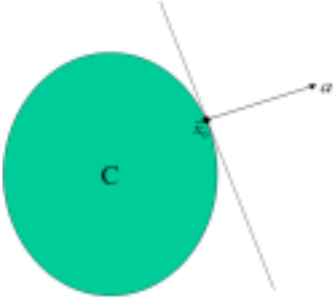
---

which means $\bar{\mathbf{a}}^T \mathbf{x} \geq \bar{\mathbf{a}}^T \mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$, and $\mathbf{y} \in \mathcal{D}$.

Supporting hyperplane theorem (consequence of separating hyperplane theorem)

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.