Linear (in)equalities, Linear programming
Conic programming

1) Recall that if S = linear vector space $\subseteq V$ & B is its basis, $S = \text{linear span}(B) = \{v \in V \mid \langle v, b\rangle_V = 0 \ \forall\, b \in B^\perp\}$
where $B^\perp$ is basis for $S^\perp$

Assuming V is an inner product space

eg: if $V = \mathbb{R}^n$ & $\langle a, b\rangle = a^T b$

$$\{v \in \mathbb{R}^n \mid \langle v, b\rangle_V = 0 \ \forall\, b \in B^\perp\}$$

$$\text{(can be written as)} \equiv \{v \in \mathbb{R}^n \mid Pv = 0\}$$

s.t. $\text{rank}(P) + \dim(S) = n$

2) Recall that if A = affine set $\subseteq V$ & B is its basis, $A = \text{affine span}(B) = \{v \in V \mid \langle v, b\rangle_V = c_b \ \forall\, b \in B^\perp\}$
where $B^\perp$ is basis for $S^\perp$ where $A = a + S$

eg: If $V = \mathbb{R}^n$ & $\langle a, b\rangle = a^T b$

$$\{v \in \mathbb{R}^n \mid \langle v, b\rangle_V = c \ \forall\, b \in B^\perp\}$$

$$\text{(can be written as)} \equiv \{v \in \mathbb{R}^n \mid Pv = c\}$$

s.t. $\text{rank}(P) + \dim(A) = n$

Q: What about dual representations of conic sets?

We will motivate through linear programming (LP) dual of LP & generalised inequalities:

$$\boxed{\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^T x \\ \text{subject to} & -Ax+b \leq 0 \end{array}}$$

→ LINEAR PROGRAM

(Recall applet: http://www.olivierdubois.info/projects/CompGeo/project/APPLET/applet.html)

Can be rewritten as $Ax \geq b$ or $Ax - b \in \mathbb{R}^n_+$

Note: $\mathbb{R}^n_+$ is a CONE. How abt defining generalised inequality for a cone K as: $c \geq_K d$ iff $c - d \in K$ and a general conic program as:

$$\boxed{\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^T x \\ \text{subject to} & -Ax+b \leq_K 0 \end{array}}$$

→ CONIC PROGRAM

That is, $Ax - b \in K$

Generalized inequalities

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

→ Also referred to as a regular cone

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Some restrictions on K that we will require. H/w: WHY?

i.e. K has no st. lines passing thru O

i.e. if $a, -a \in K$, then $a = 0$

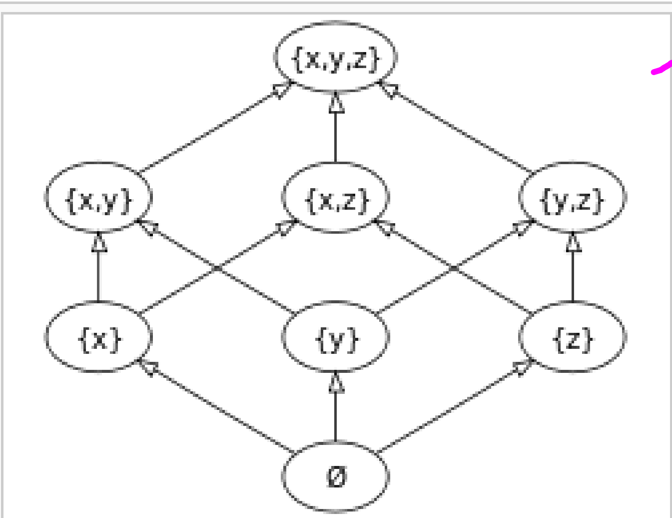
examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

To prove that K being closed, solid & pointed are necessary & sufficient conditions for $\succeq_K$ to be a valid inequality, recall that any partial order $\geq$ should satisfy the following properties (refer page 51 of www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf)

1. *Reflexivity:* $a \geq a$;
2. *Anti-symmetry:* if both $a \geq b$ and $b \geq a$, then $a = b$;
3. *Transitivity:* if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. *Compatibility with linear operations:*
   (a) *Homogeneity:* if $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$ ("One can multiply both sides of an inequality by a nonnegative real")
   (b) *Additivity:* if both $a \geq b$ and $c \geq d$, then $a + c \geq b + d$ ("One can add two inequalities of the same sign").



The Hasse diagram of the set of all subsets of a three-element set {x, y, z}, ordered by inclusion.

→ example partial order $\subseteq$ over sets

(source: http://en.wikipedia.org/wiki/Partially_ordered_set)

that is, the $\subseteq$ partial order

generalized inequality defined by a proper cone K:

$$x \preceq_K y \iff y - x \in K, \qquad x \prec_K y \iff y - x \in \mathbf{int}\, K$$

examples

- componentwise inequality ($K = \mathbf{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \iff x_i \le y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\le$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$

Minimum and minimal elements

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

$$y \in S \implies x \preceq_K y$$

$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \implies y = x$$

example ($K = \mathbf{R}^2_+$)

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2