Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum\limits_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_\infty$, $M_N(A) = \max\limits_i \sum\limits_{j=1}^{m} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$

$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation:

Absolute value of sum
is <= sum of absolute values

$N = \|.\|_1, \; M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

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2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then

C is max sum over absolute values in a column

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3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0....1.....0]$

$N = \|.\|_1$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

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4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then

All inequalities mentioned above become equalities

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5. Thus, there exists $\mathbf{x} = [0, 0..1, 0...0]$ for which the inequalities in steps (2) and (3) become equalities! That is,

$N = \|.\|_1$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

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$$M_N(A) = \|A\mathbf{x}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

H/w: Complete similar proof for infinity norm

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[8]): A^T A is always positive semi-definite

[8] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

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2. (From basic notes on Linear Algebra[8]): $A^TA \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition, applied to positive semi-definite matrix A^TA:

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4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} =$ linear combination of the ui's (basis)

[8] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

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4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$
6. Then, $\|\mathbf{x}\|_2 = \sqrt{\sum_i \alpha_i^2}$ and $\|A\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T(A^T A\mathbf{x})} =$

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6. Then, $\|\mathbf{x}\|_2 = \sqrt{\sum_i \alpha_i^2}$ and $\|A\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T(A^T A\mathbf{x})} = \sqrt{(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i)^T (\sum_{i=1}^{n} \sigma_i \alpha_i \mathbf{u}_i)}$.
7. If $\alpha_1 = 1$ and $\alpha_j = 0$ for all $j \neq 1$, the maximum value in (7) will be attained. Thus, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$

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Norm balls: Summary

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set.
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
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- If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$ inner prod?
- Matrix norm with an inner product:

Trivial extension of the vector inner product
by unfolding a matrix into a vector

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- Matrix norm with an inner product:

$$\langle A, B \rangle = \sqrt{\sum_{i,j} a_{ij} b_{ij}} =$$ trace(A^TB)

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- Matrix norm with an inner product:

$\langle A, B \rangle = \sqrt{\sum\limits_{i,j} a_{ij} b_{ij}} = \sqrt{trace(A^T B)}$ is the Frobenius inner product.

$\|A\|_F = \sqrt{\sum\limits_{i,j} a_{ij}^2} = \sqrt{trace(A^T A)}$ is the Frobenius norm.

Examples of Convex Cones

More on Convex Sets and Cones

- Half-spaces as cones (induced by hyperplanes) - as affine shifted convex cones (already discussed)
- Norm Cones
- Positive Semi-definite cone.
- Positive Semi-definite cone: Example and Notes.
- Convexity Preserving Operations on Sets

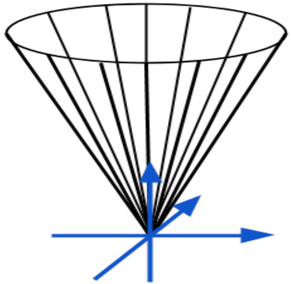
Norm cones

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius r**: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$.
- **Norm cone:** A **set** of form: obtained by stacking norm balls below each other with diminishing radius r

{ (x,z) | ||x|| <= tz}

Norm cones

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius r**: $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$.
- **Norm cone:** A **set** of form: $\{(\mathbf{x}, t) \in \Re^{n+1} \mid \|\mathbf{x}\| \leq t\}$. Canonically just a t
  - Norm cones are convex cones
  - Euclidean norm cone is called-second order cone. If $\mathbf{x} \in \Re^2$, in $\Re^3$ it appears as:



Positive semidefinite cone: **Primal Description**

Can we visualize using a Dual Description?
Can Frobenius inner product come to rescue?
v^TXv = <vv^T,X>

Notation

- S^n is set of symmetric $n \times n$ matrices.
- $S^n_+ = \{X \in S^n | X \succeq 0\}$: set of $n \times n$ positive semidefinite matrices.
  - $X \in S^n_+ \iff \mathbf{v}^T X \mathbf{v} \geq 0$ for all $\mathbf{v} \in \Re^n$
  - S^n_+ is a convex cone.
- $S^n_{++} = \{X \in S^n \mid X \succ 0\}$: set of $n \times n$ positive definite matrices.

Not a cone since 0 combinations
are not contained

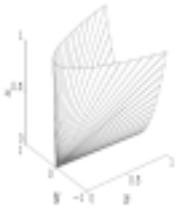
Positive semidefinite cone: **Primal Description**

Consider a positive semi-definite matrix $S \in \Re^2$. Then S must be of the form

$$S = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \quad (35)$$

Canonical representation of a symmetric positive semi-definite matrix

We can represent the space of matrices $\mathcal{S}_+^2$ in $\Re^3$ with non-negative x, y and z coordinates and a non-negative determinant:



Positive semidefinite cone: **Dual Description**

Instead of all vectors $\mathbf{v} \in \Re^n$, we can, without loss of generality, only require the inequality to hold for all $\mathbf{v}$ with $\|\mathbf{v}\|_2 = 1$.

1. $S_+^n = \{A \in S^n | A \succeq 0\} = \{A \in S^n | \mathbf{v}^T A \mathbf{v} \geq 0, \forall \|\mathbf{v}\|_2 = 1\}$
2. Note: $\mathbf{v}^T A \mathbf{v} = \sum_i \sum_j v_i a_{ij} v_j = \sum_i \sum_j (v_i v_j) a_{ij} =$ Frobenius inner product of vv^T with A

Positive semidefinite cone: **Dual Description**

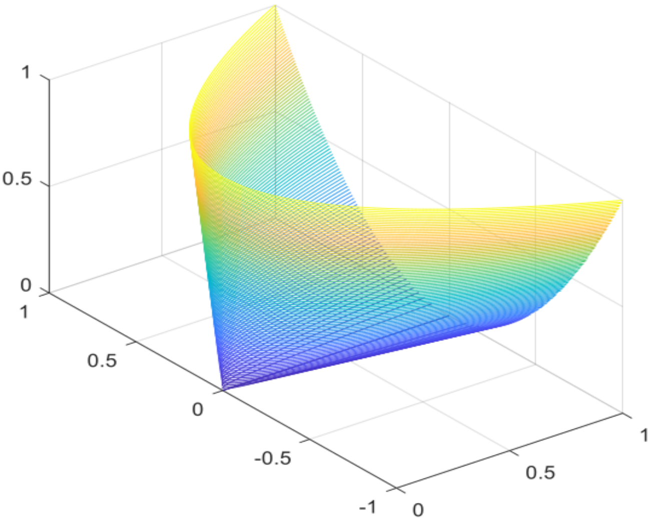
Instead of all vectors $\mathbf{v} \in \Re^n$, we can, without loss of generality, only require the inequality to hold for all $\mathbf{v}$ with $\|\mathbf{v}\|_2 = 1$.

1. $S_+^n = \{A \in S^n | A \succeq 0\} = \{A \in S^n | \mathbf{v}^T A \mathbf{v} \geq 0, \forall \|\mathbf{v}\|_2 = 1\}$
2. Note: $\mathbf{v}^T A \mathbf{v} = \sum_i \sum_j v_i a_{ij} v_j = \sum_i \sum_j (v_i v_j) a_{ij} = \langle \mathbf{v}\mathbf{v}^T, A \rangle = tr((\mathbf{v}\mathbf{v}^T)^T A) = tr(\mathbf{v}\mathbf{v}^T A)$
3. So, $S_+^n = \bigcap_{\|\mathbf{v}\|=1} \{A \in S | \langle \mathbf{v}\mathbf{v}^T, A \rangle \geq 0\}$
   - One parametrization for $\mathbf{v}$ such that $\|\mathbf{v}\|_2 = 1$ is

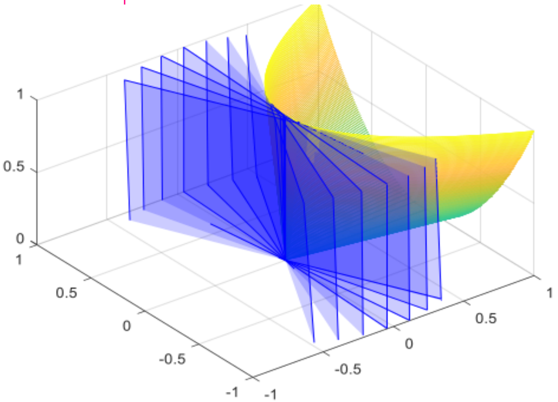
$$\mathbf{v} = \begin{bmatrix} Cos(\theta) \\ Sin(\theta) \end{bmatrix} \tag{36}$$

$$\mathbf{v}\mathbf{v}^T = \begin{bmatrix} Cos^2(\theta) & Cos(\theta)Sin(\theta) \\ Cos(\theta)Sin(\theta) & Sin^2(\theta) \end{bmatrix} \tag{37}$$

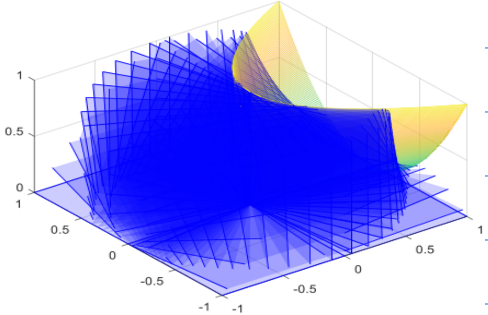
   - Homework: Plot a finite # of halfspaces parameterized by (θ).



1
0.5
0
1
0.5
0
-0.5
-1
0
0.5
1



Each hyperplane has been generated programmatically using a different value of theta



Positive semidefinite cone: **Dual Description**

1. S_+^n = intersection of infinite # of half spaces belonging to $R^{n(n+1)/2}$ [Dual Representation]
    1. Cone boundary consists of all singular p.s.d. matrices having at-least one 0 eigenvalue.
    2. Origin = O = matrix with all 0 eigenvalues.
    3. Interior consists of all full rank matrices A (rank $A = m$) i.e. $A \succ 0$.

Convexity preserving operations

In practice if you want to establish the convexity of a set $\mathcal{C}$, you could either

1. prove it from first principles, *i.e.*, using the definition of convexity or eg: norm ball
2. prove that $\mathcal{C}$ can be built from simpler convex sets through some basic operations which preserve convexity.

Some of the important operations that preserve complexity are:

1. Addition (recap discussion in context of Separating Hyperplanes)
2. Intersection
3. Affine Transform (Eg: Ellipsoid as a transform of sphere)
4. Perspective and Linear Fractional Function

Closure under Intersection

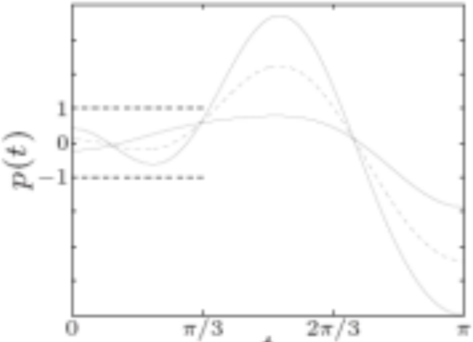
The intersection of any number of convex sets is convex. Consider the set $\mathcal{S}$:

$$\mathcal{S} = \left\{ \mathbf{x} \in \Re^n \mid |p(t)| \leq 1 \;\; for \;\; |t| \leq \frac{\pi}{3} \right\} \tag{38}$$

where

$$p(t) = x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \tag{39}$$

= <x,cos_vec(t)>



Closure under Intersection (contd.)

Any value of t that satisfies $|p(t)| \leq 1$, defines two regions, *viz.*,

$$\Re^{\leq}(t) = \left\{\mathbf{x} \mid x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \leq 1\right\}$$

and

$$\Re^{\geq}(t) = \left\{\mathbf{x} \mid x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \geq -1\right\}$$

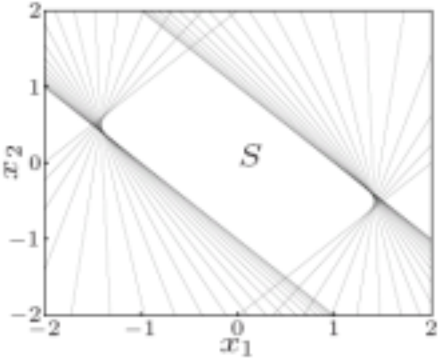
Each of the these regions is convex and for a given value of t, the set of points that may lie in $\mathcal{S}$ is given by $\Re(t) = \Re^{\leq}(t) \cap \Re^{\geq}(t)$

Intersection over intersection of halfspaces ==> Convex

Closure under Intersection (contd.)

$\Re(t)$ is also convex. However, not all the points in $\Re(t)$ lie in $\mathcal{S}$, since the points that lie in $\mathcal{S}$ satisfy the inequalities for every value of t. Thus, $\mathcal{S}$ can be given as:

$$\mathcal{S} = \bigcap_{|t| \leq \frac{\pi}{3}} \Re(t)$$



Closure under Affine transform

An affine transformation or affine map between two vector spaces $f: \Re^n \rightarrow \Re^m$ consists of a linear transformation followed by a translation:

$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$$

where $A \in \Re^{n \times m}$ and $\mathbf{b} \in \Re^m$.
An affine transform is one that preserves (eg: when you go from sphere to ellipsoid)

1) collinearity between points?
2) ratios of distances are preserved?

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- Collinearity between points, *i.e.*, three points which lie on a line continue to be collinear after the transformation.
- Ratios of distances along a line, *i.e.*, for distinct colinear points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, $\frac{||\mathbf{p}_2 - \mathbf{p}_1||}{||\mathbf{p}_3 - \mathbf{p}_2||}$ is preserved.

Closure under Affine transform (contd.)

In the finite-dimensional case each affine transformation is given by a matrix A and a vector $\mathbf{b}$. The image and pre-image of convex sets under an affine transformation defined as

$$f(\mathbf{x}) = \sum_{i}^{n} x_i a_i + b$$

yield convex sets[9]. Here a_i is the i^{th} row of A. The following are examples of convex sets that are either images or inverse images of convex sets under affine transformations:

1. the solution set of linear matrix inequality $(A_i, B \in \mathcal{S}^m)$

$$\left\{\mathbf{x} \in \Re^n \mid x_1 A_1 + \ldots + x_n A_n \preceq B\right\}$$

is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[10]. This set is the inverse image under an affine mapping of the

H/w

[9] Exercise: Prove.

[10] The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}^n_+$.