Note a H/w problem for 3rd August:

Show that the following are vector spaces (assuming scalars come from a set S), and then answer questions that follow for each of them: Set of all matrices on S, set of all polynomials on S, set of all sequences of elements of S. (HINT: You can refer to this book for answers to most questions in this homework.) How would you understand the concepts of independence, span, basis, dimension and null space (chapter 2 of this book), eigenvalues and eigenvectors (chapter 5), inner product and orthogonality (chapter 6)? EXTRA: Now how about set of all random variables and set of all functions

Let us consider space of matrices:

$$\left\{ \begin{bmatrix} s_{11} & \cdots & s_{1m} \\ & & \\ s_{n1} & \cdots & s_{nm} \end{bmatrix} \middle| \; s_{11} \ldots s_{nm} \in S \right\}$$ over scalars S

So far we considered $S = \mathbb{R}$

Obvious that this is a vector space (since multiplication etc are defined on S)

For simplicity, let $S = \mathbb{R}$ & let us consider a norms for matrices, induced by norms for vectors

Let $N(x)$ be a vector norm satisfying the vector norm axioms:

Basis for vector space of matrices $(m \times n)$

$$\left\{ \underbrace{\begin{bmatrix} 1 & 0 & \cdot & 0 \\ \cdot & & & \\ \vdots & & & \\ 0 & \cdots & \cdot & 0 \end{bmatrix}}_{E_{11}} \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \cdots \\ & & & \\ 0 & & \cdot\ - & 0 \end{bmatrix}}_{E_{12}} \cdots \underbrace{\begin{bmatrix} 0 & - & \cdot & 0 \\ \cdot & & & \\ \vdots & & & \\ 0 & \cdot & - & \cdot 1 \end{bmatrix}}_{E_{mn}} \right\}$$

$m \times n$ linearly independent elements that span the space of all matrices of size $m \times n$

$$B = \sum_{i,j} a_{ij} E_{ij} \equiv \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{mn} \end{bmatrix}$$

This vector $\in \mathbb{R}^{mn} / S^{mn}$ is a canonical representation of B

Then we will define a matrix norm

$$M_N(A) = \sup_{x \neq 0} \frac{N(Ax)}{N(x)}$$

Can you prove that this is indeed a valid norm?

$\sup_{s \in S} f(s) = \hat{f}$ if $\hat{f}$ is minimum upper bnd

as the matrix norm induced by vector norm $N(x)$

what, for example, will be $M_N(I)$ → Ans: 1

irrespective of $N(x)$?

$$\left(\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right)$$

examples

a) If $N(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$

$\|A\|_1 = ?$

Ans: $$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}| |x_j|$$

abs value of sum ≤ sum of abs values

Changing order of summation:

$$\|Ax\|_1 \leq \sum_{j=1}^{m} |x_j| \sum_{i=1}^{n} |a_{ij}|$$

→ Let $C = \max_j \sum_{i=1}^{n} |a_{ij}|$

Then $\|Ax\|_1 \le C\|x\|_1$

$$\Rightarrow \quad \|A\|_1 = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \le C$$

But consider an $x = [0 . 0 . 1 . . 0\ 0]$

k^{th} position, where k is column index j for which $C = \sum_{i=1}^{n} |a_{ik}|$

Then $\|x\|_1 = 1$ & $\|Ax\|_1 = C$ (Show this)

$$\Rightarrow \quad \|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

i.e if $N(x) = \|x\|_1$ then $M_N(A) = \max_j \sum_i |a_{ij}|$

(b) Similarly,

if $N(x) = \|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$

$$\|A\|_2 = [\text{dominant eigenvalue of } A^T A]^{1/2}$$

(c) If $N(x) = \|x\|_\infty = \max_i |x_i|$ $\left\{\lim_{p\to\infty} \left(\sum |x_i|^p\right)^{1/p}\right.$

$$\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$$

Proof for (b)

$\sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|}$ We know: $\sqrt{\langle Ax, Ax \rangle} = \|Ax\|_2$

$(Ax)^T(Ax) = x^T A^T A x$

Assume all reals for complex you use conjugate transpose

A^TA is symmetric & p.s.d ($\in S^n_+$)

$\Rightarrow$ By spectral decomposition $\exists$ orthonormal U & diagonal matrix of +ve eigenvalues of A^TA s.t $A^TA = U^T \Lambda U$ & u_i is s.t $(A^TA)u_i = \lambda_i u_i$ (without loss of generality, let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$)

Further let $x = \sum_{i=1}^{n} \alpha_i u_i$ (since columns of U form basis for $\mathbb{R}^n$)

$\langle Ax, Ax \rangle = (\sum \alpha_i u_i)^T (\sum \alpha_i \lambda_i u_i) = \sum \alpha_i^2 \lambda_i \in [\lambda_1 \sum \alpha_i^2, \lambda_n \sum \alpha_i^2]$

$\Rightarrow \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\lambda_{max}(A^TA)}$

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

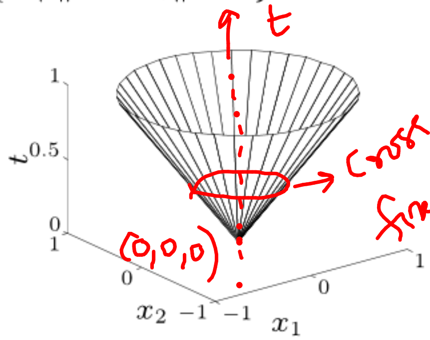
- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{(x, t) \mid \|x\| \leq t\}$ → pointedness

Euclidean norm cone is called second-order cone



Cross section fixed t has giving you a cross section as norm ball in $\mathbb{R}^2$

norm balls and cones are convex

$\|x\|_q \leq \|x\|_p \leq n^{1/p - 1/q} \|x\|_q \quad \forall \ 1 \leq p \leq q \leq \infty$

Other matrix norms:

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$$

Frobenius norm

Q: What abt inner products:

Note: Not all normed spaces are inner prod spaces.

Eg: $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$

for $p=2$ $\langle x,y\rangle = \sum_i x_i y_i$

For $p=1$ or ∞, No corresp. inner products

Read more on

http://www.math.ucsd.edu/~njw/Teaching/Math271C/Lecture_04.pdf

Eg of Frobenius inner product:

$$\langle A,B\rangle = \sum_i \sum_j a_{ij} b_{ij}$$

$$\langle A,B\rangle_W = \sum_i \sum_j a_{ij}\, b_{ij} w_{ij} \quad \text{for } w_{ij} > 0$$

weighted inner product