Convex optimization with inequality constraints: Kelly's Cutting Plane Algorithm

Consider the general convex optimization problem[1]:

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & g_i(\mathbf{x}) \leq 0 \quad \text{for } i = 1, 2, \ldots, m \end{array} \tag{1}$$

where $g_i(\mathbf{x})$ are convex functions.

Below, we reproduce the Kelly's cutting plane algorithm (the motivation for each step was discussed in class).

[1] As discussed in class, all convex optimization problems of the form discussed so far can be cast in this form.

Step 1
Input an initial feasible point, $\mathbf{x}^0$ and set $k = 0$.
Step 2
Evaluate

$$A^k = \begin{bmatrix} A_0 \\ A_1 \\ . \\ . \\ A_k \end{bmatrix} \quad \mathbf{b}^k = \begin{bmatrix} A_0\mathbf{x}^0 + \mathbf{g}_0 \\ A_1\mathbf{x}^1 + \mathbf{g}_1 \\ . \\ . \\ A_k\mathbf{x}^k + \mathbf{g}_k \end{bmatrix} \tag{2}$$

where,

$$A_i = \begin{bmatrix} \mathbf{s}_1(\mathbf{x}^i) \\ \mathbf{s}_2(\mathbf{x}^i) \\ . \\ . \\ \mathbf{s}_m(\mathbf{x}^i) \end{bmatrix} \quad \mathbf{g}_i = \begin{bmatrix} g_1(\mathbf{x}^i) \\ g_2(\mathbf{x}^i) \\ . \\ . \\ g_m(\mathbf{x}^i) \end{bmatrix} \tag{3}$$

where $\mathbf{s}_j(\mathbf{x}^i)$ is a subgradient of g_j at the point $\mathbf{x}^i$. Remember[a] every gradient is a subgradient.
Step 3
Solve the LP problem

$$\begin{aligned} \mathbf{x}_*^k = \quad & \underset{\mathbf{x}}{\text{argmin}} && \mathbf{c}^T\mathbf{x} \\ & \text{subject to} && A^k\mathbf{x} \geq \mathbf{b}^k \end{aligned}$$

Step 4
If $\max\{g_j(\mathbf{x}_*^k),\ 1 \leq j \leq m\} \leq \epsilon$ output $\mathbf{x}_* = \mathbf{x}_*^k$ as the point of optimality and stop. Otherwise, set $k = k+1$, $\mathbf{x}^{k+1} = \mathbf{x}_*^k$, update A^k and $\mathbf{b}^k$ from (2) using (3) and repeat from **Step 3.**

[a] Recall that we are only dealing with convex functions.

Figure 1: Optimization for the convex problem in (1) using Kelly's cutting plane algorithm.