1. Let $f : \Re^n \to \Re$ be a convex function and let $\mathcal{C} \subseteq \Re^n$ be such that the following set

$$C_\delta = \{\mathbf{z} \mid ||\mathbf{z} - \mathbf{x}|| \le \delta, \ for\ some\ \mathbf{x} \in closure(\mathcal{C})\}$$

is compact[1], that is, C_δ is closed and bounded, for all $\delta \ge 0$. Show that f is Lipschitz continuous on $\mathcal{C}$, i.e., there exists $L > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| \le L||\mathbf{x} - \mathbf{y}||, \ \forall x, y \in \mathcal{C}$$

State the value for L.

Hint: Use the Weierstrass' theorem, which implies that a continuous function $f : \Re^n \to \Re$ over a compact set $\mathcal{A} \subseteq \Re^n$ attains extreme values on $\mathcal{A}$.

(7 Marks)

Ans: (1) As discussed in class, a convex fn on $\mathbb{R}^n$ is continuous. Thus, f is continuous

(2) Let $p, q \in C_\delta$ then

$$r = q + \frac{\delta}{||q-p||}(q-p) \in C_\delta \quad \text{(By definition of } C_\delta\text{)}$$

$$\Rightarrow q = \underbrace{\frac{||q-p||}{||q-p||+\delta}}_{\alpha} r + \underbrace{\frac{\delta}{||q-p||+\delta}}_{1-\alpha} p$$

Thus, q is a convex combination of p & r

Since f is convex:

$$f(q) \le \underbrace{\frac{||q-p||}{||q-p||+\delta}}_{\alpha} f(r) + \underbrace{\frac{\delta}{||q-p||+\delta}}_{1-\alpha} f(p)$$

$$\Rightarrow f(q) - f(p) \le \frac{\|q-p\|}{\|q-p\| + \delta}\left(f(r) - f(p)\right)$$

$$\frac{\|q-p\|}{\|q-p\| + \delta} \le \frac{\|q-p\|}{\delta}$$

[By Weierstrass' thm, f is cts & since C_δ is closed & bounded, f attains extrema on C_δ]

$$\le \frac{\|q-p\|}{\delta}\left(\max_{z\in C_\delta} f(z) - \min_{x\in C_\delta} f(x)\right) \quad (1)$$

Similarly, we get

$$f(p) - f(q) \le \frac{\|p-q\|}{\delta}\left(\max_{z\in C_\delta} f(z) - \min_{x\in C_\delta} f(x)\right) \quad (2)$$

Combining (1) & (2), we get

$$|f(p) - f(q)| \le L\|p-q\|$$

where $L = \left(\max_{z\in C_\delta} f(z) - \min_{x\in C_\delta} f(x)\right)\Big/\delta$

2. As discussed in the class, directon $\mathbf{d}$ is a descent direction of a function f at a point $\mathbf{x}$ if the directional derivative of f along $\mathbf{d}$ is strictly negative. That is $\mathbf{d}^T \nabla f(\mathbf{x}) < 0$. In this exercise, we provide a method for generating descent direction in cases in which obtaining a single subgradient is relatively simple.

   (a) Let $\mathbf{g}_f^{(i)}(\mathbf{x})$ be a subgradient of f at $\mathbf{x}$ in the i^{th} step of the algorithm. (For $i = 0$, you just pick any subgradient.) Let $\mathbf{w}_k$ be the vector of minimum p-norm (for any $p \geq 1$) in the convex hull of $\mathbf{g}_f^{(1)}(\mathbf{x}), \mathbf{g}_f^{(2)}(\mathbf{x}), \ldots \mathbf{g}_f^{(k-1)}(\mathbf{x})$. Present an algorithm for computing $\mathbf{w}_k$ when $p = 2$. What about the case of any other value of p?

   (4 Marks)

$\ldots \mathbf{g}_f(\mathbf{x}^{(k-1)})$.

Ans: $W_k = \min_{\alpha, w} \|w\|_p$

s.t. $w = \alpha_1 g_f^{(1)}(x) + \alpha_2 g_f^{(2)}(x) \cdots + \alpha_{k-1} g_f^{(k-1)}(x)$

(b) Stop if $-\mathbf{w}_k$ is a descent direction of f at $\mathbf{x}$. Since f may not be differentiable, the criterion for $-\mathbf{w}_k$ being a descent direction of f at $\mathbf{x}$ is obtained by replacing $\nabla f(\mathbf{x})$ with $\mathbf{g}_f(\mathbf{x})$:

$$-\mathbf{w}_k^T \mathbf{g}_f(\mathbf{x}) < 0$$

If the stopping criterion is not met, let $\mathbf{g}_f^{(k)}(\mathbf{x}) \in \partial f$ such that

$$w_k^T \mathbf{g}_f^{(k)}(\mathbf{x}) = \min_{\mathbf{g}\in\partial f} w_k^T \mathbf{g}$$

Prove that this process returns a descent direction of f at $\mathbf{x}$ in a finite number of iterations. You can assume that ∂f is compact. (Note that since $\partial f \subseteq \Re^n$, this is equivalent to saying that ∂f is closed and bounded).

(7 Marks)

Ans: Firstly, w_k is the projection of the origin on the set $\text{conv}(g_f^{(1)}(x), g_f^{(2)}(x), \ldots g_f^{(k-1)}(x))$

$\therefore$ By the projection theorem on slide 9 of http://www.cse.iitb.ac.in/~CS709/notes/eNotes/first-order-descent-projectionMethod-annotated.pdf, we have $\forall\, g \in \text{conv}(g_f^{(1)}(x) \ldots g_f^{(k-1)}(x))$

$$(g - w_k)^T (0 - w_k) \leq 0$$

$$\Rightarrow (g - w_k)^T w_k \geq 0 \Rightarrow g^T w_k \geq \|w_k\|^2 \geq \underbrace{\min_{g\in\partial f(x)} \|g\|^2 = \|g_*\|^2}$$

Both $(g_f^{(1)} \ldots g_f^{(k-1)})$ & $(w_1 \ldots w_{k-1})$ are sequences lying in ∂f

Note: $w_k \in \partial f(x)$

& $g_* \in \partial f(x)$ is subgradient with minimum norm

$\Rightarrow$ Since ∂f is closed, limit points of $w_1 \ldots w_{k-1}$ & of $(g_f^{(1)} \ldots g_f^{(k-1)})$ should both lie in ∂f

$\|g_x\| > 0$, since if $g_x = 0$ then x is already an optimal point

$\therefore$ if $\lim_{k\to\infty} w_k = \hat{w}$ & $\lim_{k\to\infty} g_f^{(k)} = \hat{g}_f$ then we expect $\hat{g}_f^T \hat{w} > 0$

But if none of w_k's are descent directions so far, we have

$$-w_k^T g_f^{(k)} = -\min_{g\in\partial f} w_k^T g = \max_{g\in\partial f} -w_k^T g \geq 0$$

ie $w_k^T g_f^{(k)} \leq 0$

$\hookrightarrow$ Letting $k\to\infty$, we get a contradiction

$\hat{w}^T \hat{g}_f \leq 0$

$\Rightarrow$ Some w_k should become a descent direction

3. Consider the equality constrained optimization problem in (1):

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} + \beta \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \end{array} \tag{1}$$

Assume that A has full row rank (that is no equality is redundant/conflicting). Let N be the basis for the null space of A. Show that this optimization problem is unbounded below if $N^T Q N$ has negative eigenvalues.

(5 Marks)

Ans: Let us write KKT conditions for this problem

$$Qx^* + c + A^T\lambda^* = 0$$

Frictional loss thru a horizontal pipeline

For $L = 3000$ ft $\simeq 1$ km

$$H = \frac{4FLV^2}{2gD}$$

V: velocity needs to be estimated in ft/min

H: Delivery head

g: 9.8 m/s²

D: Diameter

HP reqd = $\frac{WQH}{75}$

W: Density of water 1000 kg/m³

Q: Flow rate of water m³/s ... in terms of buckets per hr.

in m³/s

4000 litres per hr

= 1.2 litres/sec

$\simeq \frac{1}{2}$ HP $<$ 37.5 kg m/s