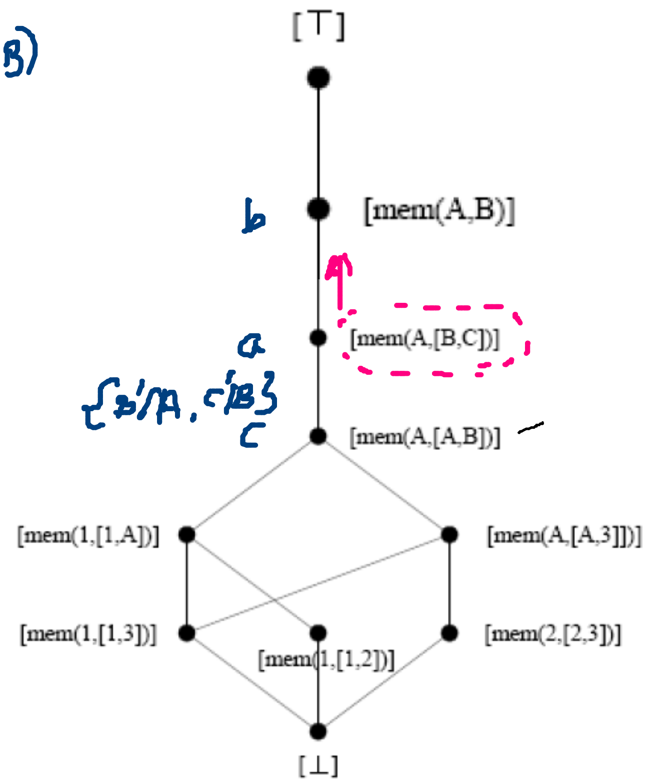
Lub	Definition	Examples
lub of terms $lub(t_1, t_2)$	1. $lub(t, t) = t$, 2. $lub(f(s1, \ldots, s_n), f(t_1, \ldots, t_n)) = f(lub(s_1, t_1), \ldots, lub(s_n, t_n))$, 3. $lub(f(s_1, \ldots, s_m), g(t_1, \ldots, t_n)) = V$, where $f \neq g$, and V is a variable which represents $lub(f(s_1, \ldots, s_m), g(t_1, \ldots, t_n))$, 4. $lub(s, t) = V$, where $s \neq t$ and at least one of s and t is a variable; in this case, V is a variable which represents $lub(s, t)$.	• $lub([a, b, c], [a, c, d]) = [a, X, Y]$. • $lub(f(a, a), f(b, b)) = f(lub(a, b), lub(a, b)) = f(V, V)$ where V stands for $lub(a, b)$. • When computing lggs lub one must be careful to use the same variable for multiple occurrences of the lubs of subterms, *i.e.*, $lub(a, b)$ in this example. This holds for lubs of terms, atoms and clauses alike.
lub of atoms $lub(\mathbf{a}_1, \mathbf{a}_2)$	1. $lub(P(s_1, \ldots, s_n), P(t_1, \ldots, t_n)) = P(lub(s_1, t_1), \ldots, lub(s_n, t_n))$, if atoms have the same predicate symbol P, 2. $lub(P(s_1, \ldots, s_m), Q(t_1, \ldots, t_n))$ is undefined if $P \neq Q$.	
lub of literals $lub(\mathbf{l}_1, \mathbf{l}_2)$	1. if $\mathbf{l}_1$ and $\mathbf{l}_2$ are atoms, then $lub(\mathbf{l}_1, \mathbf{l}_2)$ is computed as defined above, 2. if both $\mathbf{l}_1$ and $\mathbf{l}_2$ are negative literals, $\mathbf{l}_1 = \overline{\mathbf{a}_1}$, $\mathbf{l}_2 = \overline{\mathbf{a}_2}$, then $lub(\mathbf{l}_1, \mathbf{l}_2) = lub(\overline{\mathbf{a}_1}, \overline{\mathbf{a}_2}) = \overline{lub(\mathbf{a}_1, \mathbf{a}_2)}$, diff signs 3. if $\mathbf{l}_1$ is a positive and $\mathbf{l}_2$ is a negative literal, or vice versa, $lub(\mathbf{l}_1, \mathbf{l}_2)$ is undefined.	• $lub(Parent(ann, mary), Parent(ann, tom)) = Parent(ann, X)$. • $lub(Parent(ann, mary), \overline{Parent(ann, tom)}) = undefined$. • $lub(Parent(ann, X), Daughter(mary, ann)) = undefined$.

$l_1 \sqcup l_2 : 3 \times 3$ $l_1 \sqcap l_2 : 3 \times 3 \Rightarrow A^{\pm}_{E}$ is a lattice

EXAMPLE LATTICE

$[A, B] = f(A, B)$



atoms

S : If $a < b$ & $\nexists\, x \in S$ s.t. $a < x < b$ then b is up. cover of a

S = all atoms in figure.

COVERS OF ATOMS [DOWNWARD]

Theorem 25 *Let* $\mathbf{l}_1$ *be a conventional atom,* f *an* n*-ary function symbol (recall that* f *can be of zero arity and therefore a constant),* z *a variable in* $\mathbf{l}_1$*, and* $x_1, \ldots, x_n$*, distinct variables not appearing in* $\mathbf{l}_1$*. Let*

1. $\theta = \{z/f(x_1, \ldots, x_n)\}$ *and*
2. $\sigma = \{x_i/x_j\}$, $i \neq j$ $\quad x_i, x_j \in l_1$

Then $\mathbf{l}_2 = \mathbf{l}_1\theta$ *and* $\mathbf{l}_3 = \mathbf{l}_1\sigma$ *are both downward covers of* $\mathbf{l}_1$*. In fact, every downward cover of* $\mathbf{l}_1$ *must of one of these two forms (note that a special instance of the first case is when the function* f *has arity* 0 *and is therefore a constant). The substitutions* θ *and* σ *are termed elementary substitutions. In ILP, these 2 operations define a "downward refinement operator"*

$$\begin{tikzcd} l_1 \arrow[d, "\theta"'] \arrow[dr, "\sigma"] & \\ l_2 & l_2' \end{tikzcd}$$

Proof: (1) Note trivially $l_1 > l_2$

$(x, ?)$

if $x \in l_1$

$x \neq z$

then $x\theta = x$

$$l_1 \xrightarrow{\nu} m \xrightarrow{\mu} l_2 \qquad \text{Suppose } l_1 \xrightarrow{\theta} l_2$$

ν --- acts only on vars in l_1

μ --- acts only on vars in m

$\Rightarrow l_1\nu\mu = l_2 = l_1\theta$

Remember θ & $\nu\mu$ should result in same # of vars & should result in equivalent atoms

Can show that $\forall x \neq z$,

μ & ν are renaming substs

Cannot unify:

If so, θ should also have unified.

If $x = z$, ν will land up being a renaming subst if fz is also a var

$\Rightarrow m \equiv l_1$ (Contradiction)

$\therefore$ Only possibility is that $z\sigma = f(y_1 \ldots y_n)$

for distinct y not appearing in ℓ_1

& that σ maps no other var to y_i

$\Rightarrow$ $\ell_1\sigma$ & ℓ_2 are variants

$\Rightarrow$ μ is a renaming subst.

$\therefore$ -- contradicting $m > \ell_2$.

Proof for part 2 :- Exercise

$$\top$$

$\theta \swarrow \quad \searrow \sigma$?

Most general atoms $= P(\underbrace{x_1, \ldots x_n}_{\text{distinct}})$

[Assymetry] Exception! - If no constants but one f/n n≥1, then no upcover for $\perp$

Upward Cover! - By reversing steps (1) & (2) in Thm 25 $\Rightarrow$ Inverse substitution $\underbrace{\text{(function of}}_{\text{"undoes"}}$ (t,p))

"undoes" → unification $\{x/y\}$ → instantiations $\{x/c\}$

$P(x, f(z), y)$
$\uparrow$
$P(x, f(c), y)$

beware!

$P(x_4, x_3, x_2)$
$\uparrow$
$P(x_1, x_1, x_2)$

$P(x_{10}, x^{new}, y_7)$
$\uparrow$
$P(x_{10}, f(x_1, \ldots x_n), y_7)$

$P(f(c)), P(f(f(c))) \ldots$ b/w inf $\downarrow \perp$

$P(x, x) \downarrow \perp$

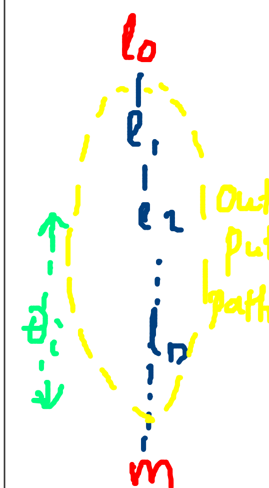
$P(f(z)) \downarrow \perp$

$P(f(x)) \to P(f(f(x))) \ldots \to$ b/w inf $\downarrow \perp$

INPUT: Conventional atoms $\mathbf{l}, \mathbf{m}$, such that $\mathbf{l} \succ \mathbf{m}$.
OUTPUT: A finite chain $\mathbf{l} = \mathbf{l}_0 \succ \mathbf{l}_1 \succ \ldots \succ \mathbf{l}_{n-1} \succ \mathbf{l}_n = \mathbf{m}$, where each $\mathbf{l}_{i+1}$ is a downward cover of $\mathbf{l}_i$.
Set $\mathbf{l}_0 = \mathbf{l}$ and $i = 0$, let θ_0 be such that $\mathbf{l}\theta_0 = \mathbf{m}$; **(1)**
if No term in θ_i contains a function or a constant; **(2) then**
  Goto 3.
else if $x/f(t_1, \ldots, t_n)$ is a binding in θ_i $(n \geq 0)$ **then**
  Choose new distinct variables $z_l, \ldots, z_n$;
  Set $\mathbf{l}_{i+1} = \mathbf{l}_i\{x/f(z_1, \ldots, z_n)\}$;
  Set $\theta_{i+1} = (\theta_i \setminus \{x/f(t_1, \ldots, t_n)\}) \cup \{z_1/t_1, \ldots, z_n/t_n\}$;
  Set i to $i+1$ and goto 2;
end if
if There are distinct variables x, y in $\mathbf{l}_i$, such that $x\theta_i = y\theta_i$ **(3) then**
  Set $\mathbf{l}_{i+1} = \mathbf{l}_i\{x/y\}$;
  Set $\theta_{i+1} = \theta_i \setminus \{x/x\theta_i\}$;
  Set i to $i+1$ and goto 3;
else if Such x, y do not exist **then**
  Set $n = i$ and stop;
end if

Handles θ (1) in thm 25

Handles 'σ' (2) in thm (25)



[finite] Downward cover chain algo.

Refinements ops : Complete set of covers
(Finite)

$\downarrow$ $\uparrow$ } clouds

1 (a) $\top$:- But it has { } as a complete set of upward.

(b) $\top$:- Yes finite complete set of downward covers

$$\top \rightarrow p(..) \quad \top \rightarrow q(..)$$

Same holds for all conventional atoms

2 (a) $\bot$:- If 'f' & no constants :- uward = { }
else if "f' & const 'c' - - upward = ∞

(b) $\bot$:- { } downward.

BACK TO CLAUSES: Subsumption thm

The Subsumption Theorem holds for first-order logic, just as it did for propositional logic:

> If Σ is a set of first-order clauses and D is a first-order clause. Then $\Sigma \models D$ if and only if D is a tautology or there is a clause C such that there is a derivation of C from Σ using resolution ($\Sigma \vdash_R C$) and C subsumes D.

Σ

$\downarrow$

C

$\subseteq$

D

$\Rightarrow$ Resolution is refutation complete for FOL.

$\therefore$ if $\Sigma \models \Box$, $\Sigma \vdash \Box$.

If $\Sigma \not\models \Box$, then does $\Sigma \nvdash \Box$ hold?

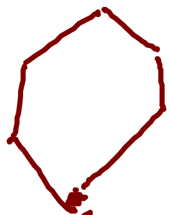
Undecidable if $\Sigma \vdash \Box$.

fns/recursive defns

$$\{mem(A,[A|B]) \leftarrow, mem(A,[B,A|C]) \leftarrow\}$$

$$\succeq$$

$$\{mem(1,[1,2]) \leftarrow, mem(2,[1,2]) \leftarrow\}$$



Equivalence class

Plotkin's reduction: $P(x) \vee Q(a) \vee P(x)$

$\equiv$

$P(x) \vee Q(a)$

obvious

$\{P(x,z), P(z,y), P(y,x)\}$

$\equiv$

$\{P(x,x)\}$.

Not so obvious!