aRb

$a \in A \qquad b \in B$

aRb

$a \in A \qquad b \in A$

Reflexive / Symmetric / Transitive

$\forall a,\ aRa$

If $aRb \Rightarrow bRa$

If aRb & if $bRc \Rightarrow aRc$

Anti-symmetric

aRb & $bRa \Rightarrow a = b$

Examples :- $\leq$ ref & Trans

$\doteq$ ref & trans

$\perp$:- symm

Function: special n-ary reln :- If $(a, b) \in f$ & $(a, c) \in f$ $\Rightarrow b = c$

R / F :- $\subseteq A \times A$

f :- fn from $A \times A \to A$

A is closed under f. (n-ary reln) if ① $f : A^n \to A$

② f is defined $\forall x \in A^n$

Algebra: A set closed under ≥ 1 operations.

$\{N, +, \times\}$

Equiv. Reln: E on set S, E partitions S into disjoint, non-empty sets

$S_1 \cup S_2 \ldots \cup S_n = S$

Proof by contradiction.

Converse: Any partition of S corresponds to an equiv. reln.

$S_a = [a]$

PARTIAL ORDERS :- $\leq$ (R) on S which is

Reflexive, anti-symm & transitive

$\leq$:-

$\subseteq$

S: Power set of T (2^T)

$\subseteq$ is a po on S.

COVERS :- If $a \leq b$ & $\nexists$ $x \in S$ s.t $a \leq x \leq b$

a is called downward cover of b

b is an upward cover of a

$T = \{a, b, c\}$ $S = 2^T = [\{a\}, \{b\} \dots]$

$\leq \equiv \subseteq$

$\underbrace{\{a, c\}}^{d.c} \leq \underbrace{\{a, b, c\}}_{u.c}$ (cover)

$\langle S, \leq \rangle$:- $\forall\, T \subseteq S$ — any arbitrary set

Least element for T → is a minimal element.

Prove unique

$a \in T,\ \forall t \in T,\ a \leq t$

None.

Not unique

Minimal element for T

$a \in T,\ \nexists\, t \in T\ \ t < a$

$\{a\}, \{b\}, \{c\}$

Lower bound of T — infimum

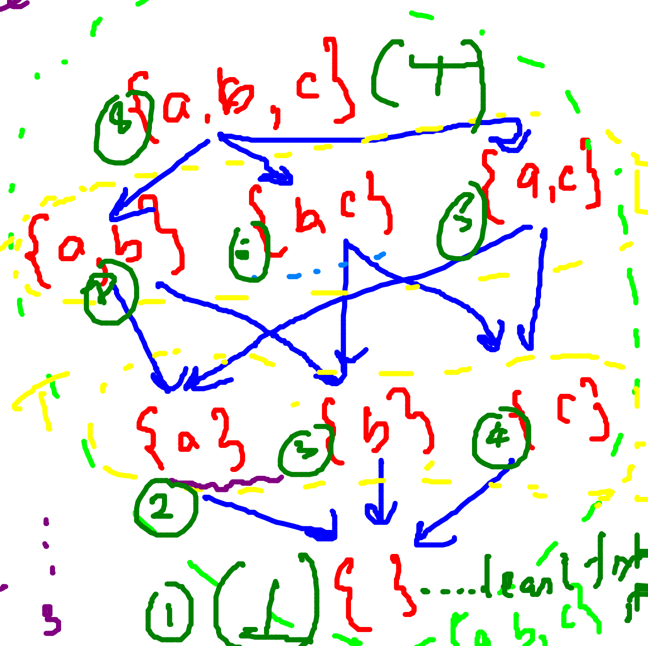
$b \in S,\ b \leq t\ \forall t \in T$

$\{\}$

GLB :- g

$b \leq g\ \forall$ lower bounds b

$\mathbb{N}$: 3, 2, 1



$S = 2^{\{a,b,c\}}$

$\leq\ =\ \subseteq$

can it have cycles?

LATTICE :- $\langle \leq, S \rangle$ s.t. for every $a, b \in S$,

least upper bound & greatest lower bnd

$\sqcup$ (LUB) $\sqcap$ (GLB)

exist

{ } lower bound for T

Restrict to $|T| = 2$

① $a \sqcap b = b \sqcap a$ . . . similarly for $\sqcup$

② $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$ . . . . ''

③ $a \sqcap (a \sqcup b) = a$ $\quad$ $a \sqcup (a \sqcap b) = a$ } absorptive

① Monotonic fn
② continuous fn } on $\langle S, \leq \rangle$

f on $\langle S, \leq \rangle$ is monotonic iff $\forall\, u, v \in S$

$\underbrace{u \leq v}_{} \Rightarrow f(u) \leq f(v)$

if u & v are comparable & $u \leq v$

$f(u) \leq f(v)$

f on $\langle S, \leq \rangle$ is cts iff $\forall$ $\underbrace{\text{directed}}_{\text{lub exists}}$ subsets

S_i of S, $f(\text{lub}(S_i)) = \text{lub}(\{f(x) \mid x \in S_i\})$

$\langle S, \leq \rangle$:- lattice

(1) Sublattice : $M \subseteq S$ is a sublattice if M is also closed under $\sqcap$ & $\sqcup$.

A lattice is an algebra under $\sqcup$ & $\sqcap$

(2) Complete lattice :- S is complete lattice if $\forall\, T \subseteq S$, glb & lub exist.

$2^{\{a,b,c\}}$:- complete lattice

Knaster – Tarski Thm :- Every monotonic fn on a complete lattice $\langle S, \leq \rangle$ ($\langle N, \leq \rangle$) has a least fixed point :- $f(x) = x$

Kleen's Recursion thm :- procedure to $s \in S$

which is least fixpoint

$s :- \perp$ $(\{\})$

$f(\perp) \leq f(f(\perp)) \leq f(f(f(\perp)))$

$[\underbrace{\{\}, \{a, b, c\} \ldots}_{\text{chain in the interval}}]$ - - - $\underbrace{f^n(\perp)}_{\text{least fixed pt}}$

maximal, minimal.