First Order Logic

$\forall x: Q(x) \leftarrow P(x)$

$\therefore \exists x\, P(x) \leftarrow R(x)$

$\Downarrow$

$\exists x\, Q(x) \leftarrow R(x)$

All humans are ~~animals~~ apes

Prop logic

H1 is an ape $\wedge$ human2 is an ape . . $\wedge$ . . .

structure: stmt abt 2 objects

Human $\subseteq$ Ape

(Statistical) Prop learners.

} Prop

① Some $\underset{R}{\text{animals}}$ are $\underset{P}{\text{humans}}$ . . .

② All $\underset{P}{\text{humans}}$ are $\underset{Q}{\text{apes}}$ . .

Therefore some $\underset{R}{\text{animals}}$ are $\underset{Q}{\text{apes}}$

③ Q

Syntax.

(1) Constants:- Object names, "lower case", ("tom") (henry)

(2) Variables:- For all X, if x is a human, then x is an ape.

(3) Quantifiers: $\forall$ $\exists$ at least one ($\neq$ some)

(4) Predicates:- Denote attributes

Human(fred) Likes(fred, banana)

Likes/2

(5) Functions:- The father of Fred is a human

Human(father(fred)). father/1

Things to note

① Variables need not denote diff objects.

$$\forall x \forall y \, Likes(x,y)$$

② $\forall x \forall y \, Likes(x,y) \equiv \forall y \forall z \, Likes(y,z)$

Variants.

③ $\forall x \forall y \, Likes(x,y) \wedge \forall x \forall y \, Hates(y,x)$

④ Order does not matter if $\forall$ or $\exists$ are exclusively used

BUT: Order does matter when $\forall$ & $\exists$ are mixed.

$$\exists x \forall y \, Likes(x,y) \not\equiv \forall y \exists x \, Likes(x,y)$$

⑤ Free variables: Not quantified

Else: Bound var.

In a single formula, some var can have Free & Bound

$\exists x [Likes(x,y) \wedge \exists y \, Dislikes(y, \ldots$

$\exists x\,(\text{Likes}(x,y) \wedge \exists y\,\text{Dislikes}(y,x))$

free.

A formula without free variables $\equiv$. SENTENCE.

Interested in truth

⑥ Negation: Beware

Some apes are not humans

(complementation)

$\checkmark\ \exists x\,(\text{Ape}(x) \Rightarrow \neg\text{Human}(x))$

$\exists x\,(\text{Ape}(x) \wedge \neg\text{Human}(x))$

It is not true that some apes are humans

True negation.

$\neg\exists x\,(\text{Ape}(x) \wedge \text{Human}(x))$

(4) Beware with $\exists$ & $\forall$

"If something has a tail, it is not an ape"

$\forall$ $\exists$ (f) $Tail(x)$ $ape(x)$

$$\forall x \, (\neg Ape(x) \leftarrow Tail(x))$$

Constants, variables, Predicate sym, Function sym (Functor), Quantifier, Logical connective, Brackets

(1) Term:- constant, variable, functional exp.

x, fred, father(fred), father(x)

(2) Atomic formula:- Predicate applied to a tuple of terms
(Atom)

Likes (fred, father(father(x)))

(3) Ground atoms:- AF w/o variables.

G(Z)

wff:

1. Any ground AF
2. If α is wff, then so is $\neg\alpha$
3. If α & β are wffs, so is $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \leftarrow \beta$
4. If α is wff containing constant c,

$\alpha^{c/x}$ ... replaces all occurrence of c in α with x.

Then $\forall x . \alpha^{c/x}$ $\exists x \alpha^{c/x}$

wffs ≡ sentences.

Clausal form

$\forall x\, (\text{Ape}(x) \leftarrow \text{Human}(x))$ $\quad\because\quad$ $(\alpha \leftarrow \beta) \equiv (\alpha \vee \neg \beta)$

$\forall x\, ((\text{Ape}(x) \vee \neg \text{Human}(x)))$

Clauses: (1) Not have quantifiers
(2) Disjunction of literals (atomic or negated atomic formulae)

$\forall x_1 \forall x_2 \ldots (\alpha_1 \wedge \alpha_2 \ldots) \longrightarrow (\alpha_1, \alpha_2 \ldots \alpha_n)$

$\alpha_1 \equiv (\beta_1 \vee \beta_2 \ldots \beta_m)$

literal.

gf(x, y) :- f(x, z), p(z, y). Prolog.

$\forall x \forall y \forall z\, (gf(x,y) \vee \neg f(x,z) \vee \neg p(z,y))$ CF

f(harry, jane).
p(jane, john).

Ground facts

Definite clause

$gf(x,y) \leftarrow f(x,z), p(z,y)$

Skolemization: Replacing existentially quantified vars with a new term: function of all previously universally quant. vars.

$\forall x_1 \ldots \forall x_n \exists y \quad \alpha$

$\underbrace{f}_{\text{Skolem fn.}}(x_1, x_2 \ldots x_n)$

$\underbrace{\forall x_1 \ldots \forall x_n}_{y \text{ bound}} \exists y \; \underbrace{\forall x_{n+1} \ldots}_{y \text{ unbound}}$

$f(\,) \equiv c$

$\exists y \forall x \, Likes(x, y)$

$\Downarrow$

$\forall x \, Likes(x, c)$

$Likes(x, c)$

(Formula α with only univ. quant.)

Remember:- $f/0 \equiv c$.

Prenex Normal Form

All quant in front.

Skolem normal form.

$\forall \bar{x} \, \alpha$

α

$\downarrow$

PNF (CPNF)

$\{\phi_1 \wedge \phi_2 \ldots\}$

$\downarrow$

SNF

PNF & disjunction.

$\downarrow$

CNF.

1. $\alpha \Rightarrow \Sigma$ (set of clauses) goal.
2. Rename vars to ensure no two vars have same name
   modulo scope

   $(\forall x, (P(x,y) \wedge \forall x\, Q(y,x)))$
3. Eliminate "$\leftarrow$" & iffs
4. Move "$\neg$" inwards

   $\neg(\exists x)\alpha \Rightarrow (\forall x)\neg\alpha$
5. $\vee$ & $\wedge$ distribute

⑤ $\forall x(\alpha_1 \wedge \alpha_2) \equiv (\forall x)\alpha_1 \wedge (\forall x)\alpha_2$.

PNF

SEMANTICS

(Recall) :- ① ~~Interp~~
② Models
③ Entailment

T/F to the props.

Subset [props]

Not so simple here. pred, fns, vars, :- Likes(fred, bananas)

① Vocab rich :-

② Quantifiers :-

$-\cdot R^{2}_{Q} :- \subseteq Q \times Q$

$p/n \subseteq X_1 \times X_2 \cdots \times X_n$

Mappings

1. constant symbols $\longrightarrow$ objects ($\mathcal{D}$)
2. predicate symbols $\longrightarrow$ Relations over objects

Structure

I :- spec of model
Suff for $T|F$ to atoms (ground)

1. Domain $\mathcal{D}$
2. Mapping of const $\rightarrow \mathcal{D}$
3. Each pred symbol $\rightarrow$ Rel on $\mathcal{D}^n$.
4. function $\rightarrow f : \mathcal{D}^n \rightarrow \mathcal{D}$

R_{Likes}

Likes(fred, banana)
O_1 O_2

Likes(fred, ban) = T iff $(O_1, O_2) \in R_{\text{likes}}$ (n ary)

Connectives :- 0 order

Quantifiers. (1) Any wff $\forall \bar{x}\, \alpha$ iff for every element in D^n, you associate with $\bar{x}$, α is true

truth of ground atom.

(2)

Models

Validity / Unsat

Consequence / Ent.

Sat

Deduction thm.

Equivalence.

(1) $\neg(\forall x)\alpha \equiv (\exists x)\neg\alpha$.

(2) $\neg(\exists x)\alpha \equiv (\forall x)\neg\alpha$.

(3) $(\forall x)\alpha \equiv (\forall y)\alpha^{x/y}$

(4) $\exists x\, \alpha \equiv (\exists y)\alpha^{x/y}$

(5) $\forall x_1 \forall x_2\, \alpha \equiv \forall x_2 \forall x_1\, \alpha$

(6) $\exists x_1 \exists x_2\, \alpha \equiv \exists x_2 \exists x_1\, \alpha$

(7) $\forall x(\alpha \wedge \beta) \equiv (\forall x\, \alpha \wedge \forall x\, \beta)$

(8) $\forall x\, \alpha \vee \forall x\, \beta \models \forall x\, (\alpha \vee \beta)$

$\exists x(\alpha \vee \beta) \equiv \exists x\, \alpha \vee \exists x\, \beta$

$\exists x(\alpha \wedge \beta) \not\models (\exists x\, \alpha \wedge \exists x\, \beta)$

$\forall x\, \alpha^{y|f(x)} \models \forall x\, \exists y\, \alpha$

$\forall x\, \alpha^{y|c} \models \exists y\, \forall x\, \alpha$

skolemization.

$M_{Skl} \subseteq M$

More on Normal forms

$$\Sigma \longrightarrow \underset{\text{PNF}}{\phi} \longrightarrow \phi^S$$

Theorem:- ϕ is satisfiable if & only if ϕ^S is satisfiable

$\phi^S \models \phi$

Prove: $\phi^S = S(S(\ldots S(\phi)))$. : Look at a single step of Skolemization.

ϕ : PNF

$S(\phi)$

rm: variable / constant

rm variable / skolem fn.

special case

Q_i be the existential quant removed $\boxed{\phi} \Rightarrow \boxed{\phi^S}$

$$\psi(x_1 \ldots x_i) = Q_{i+1} \ldots Q_n \; \phi_0(x_1 \ldots x_n)$$

$$\phi(x_1 \ldots x_n) = \forall x_1 \ldots \forall x_{i-1} \, \exists x_i \, \psi(x_1 \ldots x_i)$$

Suppose model M . c_i

$\swarrow$ Extend $f(c_1 \ldots c_{i-1}) =$

M_f

M_f is a model for $\cancel{\phi(x_1 \ldots x_n)}$ $S(\phi)$

$\phi(x_1 \ldots x_{i-1}, f(x_1 \ldots x_{i-1}), x_{i+1} \ldots x_n)$

Interpret f s.t. M_f is a model for $\psi(c_1, c_2 \ldots c_{i-1}, f(c_1 \ldots c_{i-1}))$ $\forall c_1 \ldots c_{i-1} \in M$ as values of $x_1 \ldots x_{i-1}$

$\forall x_1 \ldots x_{i-1}\, \psi(x_1, \ldots x_{i-1}, f(x_1 \ldots x_{i-1}))$ has model M!

Based on meaning of $\exists$

$\Downarrow$

$\phi_s \models \phi$

M is a model for

$\forall x_1 \ldots \forall x_{i-1} \exists x_i\, \psi(x_1 \ldots x_i)$

Obs:

1. $S(\phi) \models \phi$ . . . . . $\phi^s \models \phi$
2. ϕ sat iff ϕ^s sat.
3. $\phi \not\equiv S(\phi)$ . . . .

$\phi = \exists x\, First(x)$

$S(\phi) = First(c)$

Herbrand Ints

↓

Restricting to symbols

↳ Herbrand Univ :- U_L ≐ set of all ground (variable-free) terms that can be constructed using constants & fn symbols available in language L.

Nat(succ(y)) :- Nat(y)

Const symb: zero

Pred symb:- Nat/1

Funct symb:- pred, succ

$U_L = \{$zero, pred(zero) succ(zero), pred(succ(zero)), succ(pred(zero)) - - - - - - - $\}$

(HBase) $B_L = \{ \text{pred}(U_L) = \{$Nat(zero), Nat(pred(zero)) - - Nat(succ - - -$\}\}$

Herbrant I_L:- (Just like prop. considering B_L as the set of propositions)

$$I_L \subseteq B_L$$

~~An~~ M_L is an I_L that makes ϕ True!

$$I_1 = \{Nat(zero)\}$$

$$\Sigma_1 = Nat(zero) \wedge \forall x \, (Nat(succ(x)) \leftarrow Nat(x))$$

Q: Is I_1 a model for Σ_1?