Lattice of atoms

① Subsumption is a quasi-order

② ∴ Equivalence classes of atoms are related by partial order.

③ $A_1 \equiv A_2$ ... iff → same pred symbol
→ One-to-one correspondence between var names in A_1 & A_2.

$\underbrace{A_1}_{}$ [A_1] : eqn class

parent(x, y)
≡ parent(A, B)

parent(x, y) ≢ parent(sister(A), B)

A_1 ------> $\theta = \{x / sister(A),\ y / B\}$ A_2

$A_1 \succ A_2$ <------ α

Lattice structure over atoms contd.

$\mathcal{A}^+ = \mathcal{A} \cup \{\top, \bot\}$ is a quasi order s.t

- $\top \succeq \mathbf{l}$ for all $\mathbf{l} \in \mathcal{A}^+$
- $\mathbf{l} \succeq \bot$ for all $\mathbf{l} \in \mathcal{A}^+$
  conventional
- $\mathbf{l} \succeq m$ iff there is a substitution θ such that $\mathbf{l}\theta = m$, for $\mathbf{l}, \mathbf{m} \in \mathcal{A}$

$\mathcal{A}^+_{\dot{E}}$ ... partially ordered.

3?

Eg: $l = Mem(x, [x, y])$ $\quad m = Mem(1, [1, 2])$

Then -- $l \succeq m$ with $\theta = \{x/1, y/2\}$

$l = ?$ $\quad Mem[x_1, [x_1, y_1]]$

$\quad Mem[x_2, [x_2, y_2]]$

Lattice structure over atoms contd.

Q: If l & m are "atoms" then is $l \succeq m$ iff $l \models m$?

for clauses C_1 & C_2

$\underbrace{C_1 \succeq C_2 \text{ only if } C_1 \models C_2}_{\text{it did not hold because of self\_recursive clauses}}$

$\Rightarrow$ Logical implication over atoms is also a quasi-order over atoms

Lattice structure over atoms contd.

Embodies relations

glb

- $[\bot] \sqcap [\mathbf{l}] = [\bot]$, and $[\top] \sqcap [\mathbf{l}] = [\mathbf{l}]$

- If $\mathbf{l}_1, \mathbf{l}_2 \in \mathcal{A}$ have a most general unifier (see page 78) θ then $[\mathbf{l}_1] \sqcap [\mathbf{l}_2] = [\mathbf{l}_1\theta] = [\mathbf{l}_2\theta]$.

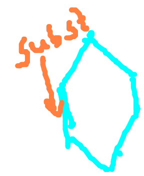
  This can be proved as follows. Let $[\mathbf{u}] \in \mathcal{A}_E^+$ such that $\boxed{[\mathbf{l}_1] \succeq [u]}$ and $[\mathbf{l}_2] \succeq [\mathbf{u}]$, then we need to show that $[\mathbf{l}_1\theta] \succeq [\mathbf{u}]$. If $[\mathbf{u}] = [\bot]$, this is obvious. If $[\mathbf{u}]$ is conventional (EA), then there are substitutions σ_1 and σ_2 such that $[\mathbf{l}_1\sigma_1] = [\mathbf{u}] = [\mathbf{l}_2\sigma 2]$. Here we can assume σ_1 only acts on variables in $\mathbf{l}_1$, and σ_2 only acts on variables in $\mathbf{l}_2$. Let $\sigma = \sigma_1 \cup \sigma_2$. Notice that σ is a unifier for $\{[\mathbf{l}_1], [\mathbf{l}_2]\}$. Since θ is an mgu for $\{[\mathbf{l}_1\sigma_1], [\mathbf{l}_2\sigma_2]\}$, there is a γ such that $\underbrace{\theta\gamma = \sigma}_{\text{By defn}}$. Now $[\mathbf{l}_1\theta\gamma] = [\mathbf{l}_1\sigma] = [\mathbf{l}_1\sigma_1] = [\mathbf{u}]$, so $[\mathbf{l}_1\theta] \succeq [\mathbf{u}]$.

  Assume l_1 & l_2 standardized apart

- If $\mathbf{l}_1, \mathbf{l}_2 \in \mathcal{A}$ do not have a most general unifier θ then $[\mathbf{l}_1] \sqcap [\mathbf{l}_2] = [\bot]$.

  Since $\mathbf{l}_1$ and $\mathbf{l}_2$ are not unifiable, there is no conventional atom $\mathbf{u}$ such that $[\mathbf{l}_1] \succeq [\mathbf{u}]$ and $[\mathbf{l}_2] \succeq [\mathbf{u}]$. Hence $[\mathbf{l}_1] \sqcap [\mathbf{l}_2] = [\bot]$.

  won't hold simultaneously



lub

- $[\bot] \sqcup [\mathbf{l}] = [\mathbf{l}]$, and $[\top] \sqcup [\mathbf{l}] = [\top]$

- If $\mathbf{l}_1$ and $\mathbf{l}_2$ have an "anti-unifier" $\mathbf{m}$ then $[\mathbf{l}_1] \sqcup [\mathbf{l}_2] = [\mathbf{m}]$; otherwise $[\mathbf{l}_1] \sqcup [\mathbf{l}_2] = [\top]$: Proof on [Q6]

Anti-unification = Reverse of unification

Idea:- Move from constants to variables.

[Term-place notation]

$Mem(1, [1,2])$

appears in 2 places

place = p

$\{(1, \langle 1 \rangle), (1, \langle 2,1 \rangle), (2, \langle 2,2 \rangle)\}$

t

t = term

$p(a, f(g(b), p(q(c)))$

$\langle t, p \rangle = ?$

1. Let $\mathbf{l} = \mathbf{l}_1$ and $\mathbf{m} = \mathbf{l}_2$, $\theta = \emptyset$, $\sigma = \emptyset$
2. If $\mathbf{l} = \mathbf{m}$ return $\mathbf{l}$ and stop.
3. Try to find terms t_1 and t_2 that have the same (leftmost) place in $\mathbf{l}$ and $\mathbf{m}$ respectively, such that $t_1 \neq t_2$ and either t_1 and t_2 begin with different function symbols, or at least one of them is a variable.

   Parent(ann, mary)
   Parent(ann, tom)
   $\Downarrow$
   Parent(ann, x)

4. If there is no such t_1, t_2, return $\mathbf{l}$ and stop.
5. Choose a variable x that does not occur in either $\mathbf{l}$ or $\mathbf{m}$ and wherever t_1 and t_2 occur in the same place (places) in $\mathbf{l}$ and $\mathbf{m}$, replace each of them by x

   p(a, a) — q(b, c)
   p(b, b)
   $\Downarrow$ p(x, x)
   $\Downarrow$ q(x, y)

6. Set θ to $\theta \cup \{x/t_1\}$ and σ to $\sigma \cup \{x/t_2\}$
7. Go to Step 3

ANTI-UNIFICATION ALGO

BEWARE: You need to take minimal step

$P(f(a), g(b))$ $\qquad$ $P(f(c), d)$

$P(f(x), g(y))$ $\qquad$ $P(x, y)$ $\qquad$ $P(f(z), y)$

? $\qquad$? $\qquad$?

Q: $Mem(1, [1,2]) \sqcup Mem(2, [2,4]) = ?$

Theorem 21 *After each iteration of the Anti-Unification Algorithm, there are terms $s_1, \ldots, s_i$ and $t_1, \ldots, t_i$ such that:*

1. $\theta = \{z_1/s_1, \ldots, z_i/s_i\}$ *and* $\sigma = \{z_1/t_1, \ldots, z_i/t_i\}$.
2. $\mathbf{l}\theta = \mathbf{l}_1$ *and* $\mathbf{m}\sigma = \mathbf{l}_2$.
3. *For every* $1 \leq j \leq i$, s_j *and* t_j *differ in their first symbol.*
4. *There are no* $1 \leq j, k \leq i$ *such that* $j \neq k$, $s_j = s_k$ *and* $t_j = t_k$.

Trivially by construction

By virtue of step 5.

Theorem 22 *Let* l_1 *and* l_2 *be two atoms with the same predicate symbol. Then the Anti- Unification Algorithm with* l_1 *and* l_2 *as inputs returns* $l_1 \sqcup l_2$.

① Algorithm will terminate after a finite # of steps (since finite terms)

② Let u be finally returned atom.

θ, σ be final substitutions

By Thm 21, $\underbrace{u\theta = l_1 \ \& \ u\sigma = l_2}_{\text{By composition}}$ $\Big\}$ $u \geq l_1$, $u \geq l_2$

③ To show:- $u = l_1 \sqcup l_2$

Lets say $v \geq l_1$ $v \geq l_2$: Show $v \geq u$

Let $w = u \sqcap v$ (exists by prev lemma)

$\underline{u \geq w}$ & $v \geq w \Rightarrow u\gamma = w$ & $v\gamma' = w$

$\Downarrow$

$w \geq l_1$ $w \geq l_2$ {By props of lattice} (else l_1 would be glb)

$$\begin{tikzcd}
 & u \arrow[dl, "\theta"'] \arrow[d, "\gamma"] \arrow[dr, "\sigma"] & \\
l_1 & w \arrow[l, "\mu"] \arrow[r, "\nu"'] & l_2
\end{tikzcd}$$

$u\gamma = w$

$l_1 = w\mu = u\gamma\mu = u\theta$

$l_2 = w\nu = u\gamma\nu = u\sigma$

$\Rightarrow$ If x is var in u

then $x\theta = x\gamma\mu$

$x\sigma = x\gamma\nu$

Claim: $w \in [u]$ ie γ is simply a renaming subst [1-1 corp]

By contradiction :- Say γ maps var "x" to non-var "t" 1 / γ unifies "z" & "y" 2

① $x\Gamma = t$ ←

→ If x is not any z_j, then

$\bar{\theta}$ does not act on x

$x\Gamma u = x\theta = x$ -- contradiction to

→ $x = z_j$ ⇒ $x\theta = s_j = x\Gamma u = tu$

$x\sigma = t_j = x\Gamma v = tv$

⇓

s_j & t_j start with first symbol of t ⇒ contradiction to thm 21 (3)

Suppose x and y are distinct variables in $\mathbf{u}$ such that γ unifies x and y. Then,

②

1. If neither x nor y is one of the z_j's, then $x\gamma\mu = x\theta = x \neq y = y\theta = y\gamma\mu$, contradicting $x\gamma = y\gamma$
2. If x equals some z_j and y does not, then $x\gamma\mu = x\theta = s_j$ and $x\gamma\nu = x\sigma = t_j$, so $x\gamma\mu \neq x\gamma\nu$ by theorem 21, part 3. But $y\gamma\mu = y\theta = y = y\sigma = y\gamma\nu$, contradicting $x\gamma = y\gamma$.
3. Similarly for the case where y equals some z_j and x does not.
4. If $x = z_j$ and $y = z_k$, then $j \neq k$, since $x \neq y$. Furthermore, $s_j = x\theta = x\gamma\mu = y\gamma\mu = y\theta = s_k$ and $t_j = x\sigma = x\gamma\nu = y\gamma\nu = y\sigma = t_k$. But this contradicts theorem 21, part 4 ↳ s_j & t_j cannot start with same symbol

↳ $i \neq j$ then $S_i = S_j$ & $t_i = t_j$ not possible