The implication order ($\models$)

Recall subsumption $A \succeq_s B$ iff $\exists\, \theta$ s.t. $A\theta \subseteq B$

$A \models B$ iff $M_A \subseteq M_B$.

Q: Will implication benefit us more?

1. It can handle relationships between recursive clauses

$$C = P(f(x)) \leftarrow P(x)$$
$$D = P(f^2(x)) \leftarrow P(x)$$

- C_1 may entail C_2 though not subsume C_2
- Two tautologies may NOT be subsume equivalent

2. Subsumption (esp for LGG) can over-generalize

$D_1 = P(f^2(a)) \leftarrow P(a)$ $\quad$ $D_2 = P(f(b)) \leftarrow P(b)$

$\{(a, \langle 1,1,1\rangle), (f(a), \langle 1,1\rangle), (f^2(a), \langle 1\rangle), (a, \langle 2\rangle)\}$

$(b, \langle 1,1\rangle), (f(b), \langle 1\rangle), (b, \langle 2\rangle)$

$C_1: P(f(a)) \leftarrow P(a)$

$C_2: P(f(b)) \leftarrow P(b)$

$\longrightarrow P(f(x)) \leftarrow P(x)$

LGS

$D_1: P(f(a)) \leftarrow P(a)$

$*D_2: P(f^2(b)) \leftarrow P(b)$

$\longrightarrow P(f(y)) \leftarrow P(x)$

LGS

More desirable: $P(f(x)) \leftarrow P(x)$

LGI

Claim that under $\geq_s$, LGI $\leq_s$ LGS

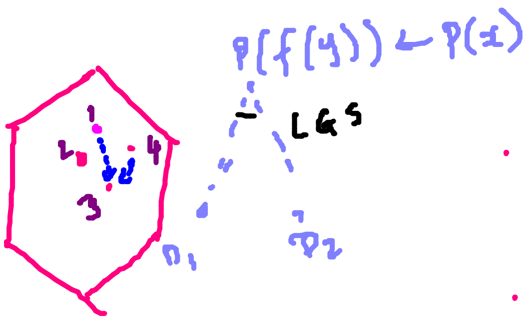
Implication is reflexive & transitive

$\Downarrow$

Quasi order

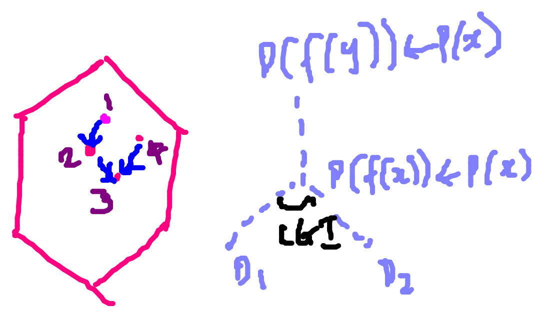
$\Downarrow$

Partial order over equivalent classes w.r.t implication



Subsumption

po



Implication

po

{ More fine-grained

Higher resolution }

Holds even for clauses without function symbols!

D_1 :- $P(x,y,z) \leftarrow P(y,z,x)$

D_2 :- $P(x,y,z) \leftarrow P(z,x,y)$

Check using resolution (cycling) that

$D_1 \models D_2$

& $D_2 \models D_1$

$\Rightarrow$ $D_1 \equiv D_2$

$LGI(D_1, D_2) = D_1 \equiv D_2$

$LGS(D_1, D_2) = ?$

Note: $D_1 \not\succeq_s D_2$ & $D_2 \not\succeq_s D_1$

$LGS(D_1, D_2) = \{ \underbrace{P(x, y, z) \leftarrow P(u, v, w)}_{\text{Overgeneralization}} \}$

Problem area :- Recursive clauses in subsumption order

$\underbrace{P}_{+}$ $\quad \neg \underbrace{P}_{-}$

③ Subsumption cannot help us compare between a theory Σ with another theory Σ' (or clause C). Implication CAN $\Big\}$ Theory refinement $\Sigma \rightarrow \Sigma'$

$\Sigma = \{ \underbrace{(P \leftarrow Q), (Q \leftarrow R)}_{\text{CNF}} \}$ $\quad C = P \leftarrow R$

$\Sigma \models C$ ... but subsumption not helpful.

LGI & GSI

→ For general formulae α_1 & α_2.

? for clauses

[sometimes used] $LGI(\alpha_1, \alpha_2) = \alpha_1 \wedge \alpha_2$ $\models \alpha_1$ $\models \alpha_2$

[hardly used] $GSI(\alpha_1, \alpha_2) = \alpha_1 \vee \alpha_2$ $\models \alpha_2$ $\models \alpha_1$

} holds even if α_1 & α_2 are clauses

$$\begin{tikzcd}
 & \phi_1 \wedge \phi_2 \arrow[dl] \arrow[dr] & \\
\phi_1 \arrow[dr] & & \phi_2 \arrow[dl] \\
 & \phi_1 \vee \phi_2 &
\end{tikzcd}$$

If ϕ_1 & ϕ_2 are clauses is $\phi_1 \vee \phi_2$ a clause?

Please standardize apart ϕ_1 & ϕ_2

LGI for clauses

↳ Exists ($\mathcal{A}$ is computable) for every finite set of clauses Σ containing at least one (1) function free, (2) non-tautologous clause

INPUT: A finite set $\mathcal{S}$ of clauses, containing at least one non-tautologous function-free clause;
OUTPUT: An LGI of $\mathcal{S}$ in $\mathcal{C}$;
Remove all tautologies from $\mathcal{S}$, call the remaining set $\mathcal{S}'$;
Let m be the number of distinct terms (including subterms) in $\mathcal{S}'$, let $V = \{x_1, \ldots x_m\}$;
f(c) c
Let $\mathcal{G}$ be the (finite) set of all clauses which can be constructed from predicate symbols and constants in $\mathcal{S}'$ and variables in V; → Do not make extra fn applications
Let $\{U_1, \ldots, U_n\}$ be the set of all subsets of $\mathcal{G}$;
Let H_i be an LGS (computed using algorithm in Figure 2.4) of U_i, for every $1 \leq i \leq n$;
Remove from $\{H_1, \ldots, H_n\}$ all clauses which do not imply $\mathcal{S}'$ (since each H_i is function-free, this implication is decidable), and standardize the remaining clauses $\{H_1, \ldots, H_q\}$ apart. ;
Provable
return $H = H_1 \cup \ldots \cup H_q$;

Note:- S' is constructed from S

If Σ = finite set of ground clauses & C is ground cl. s.t. C = L₁ ∨ ... ∨ Lₙ, A is finite set of ground atoms from Σ & C, then..

$\Sigma \models C$ iff $\Sigma \cup \{\neg L_1 \ldots \neg L_n\}$ is satisfiable

by deduction thm

iff $\Sigma \cup \{\neg L_1 \ldots \neg L_n\}$ has no herbrand model

finite check $\Rightarrow$ decidable

iff no subset of A is a H-model of $\Sigma \cup \{\neg L_1 \ldots \neg L_n\}$.

U_H (H-univ)

Negative Results

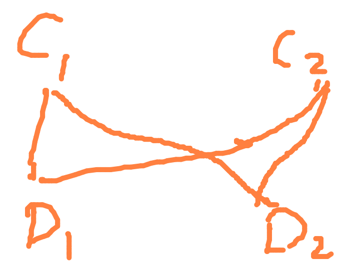
① $\exists$ pairs of horn clauses that have no LGI in $\mathcal{H}$.

$$D_1 = P(f^2(x)) \leftarrow P(x)$$

$$D_2 = P(f^3(x)) \leftarrow P(x)$$

$$C_1 = P(f(x)) \leftarrow P(x)$$

$$C_2 = P(f^2(y)) \leftarrow P(x)$$



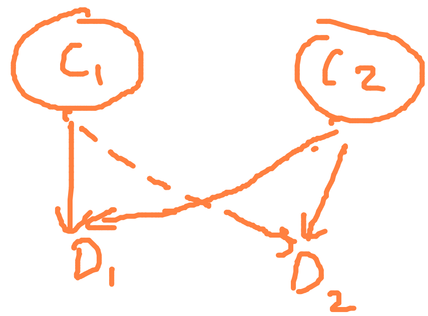
But $\underbrace{\text{LGI}}(D_1, D_2)$ exists in C

$$= P(f(x)) \vee P(f^2(y)) \vee \neg P(x)$$

② $\exists$ pairs of horn clauses with no GSI in $\mathcal{H}$.

$D_1 = P(f^2(x)) \leftarrow P(x)$

$D_2 = P(f^3(x)) \leftarrow P(x)$



$C_1 = P(f(x)) \leftarrow P(x)$

$C_2 = P(f^2(y)) \leftarrow P(x)$

$\exists$ no GSI(C_1, C_2)

③ for general clause all having fn symbols,

LGI is open question

existence computation

④ Covers --- -ve results carry over . . .

$\{P(x_1, x_2)\}$ has no upward cover.

$\{P(x_1, x_2), P(x_2, x_1)\}$ has no finite complete set of downward covers.

Positive results

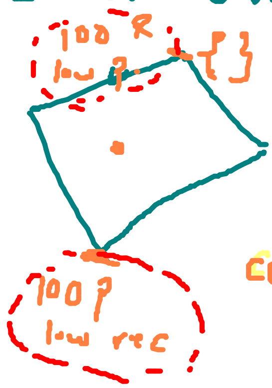
① If C is a set of function-free clauses, then $\langle C, \models \rangle$ is a lattice

ILP & Structured search space : Depends on language of H, & on notion of cover $\Rightarrow$ Prof Proc.

Recall that ILP is for developing H given B s.t $P(\text{covers}(H, B, E^+))$ is high & $P(\text{covers}(H, B, E^-))$ is low

Covers so far ignored B

① Lattice structure is a generality order over the hypothesis $\Rightarrow$ gives you a search space



cover few in E^+ & few in E^-

cover many in E^+ & possibly many in E^-. Eg :- $\{\} \equiv \top \models$ everything

② To prune large parts of search space

Might require use of monotonic $\boxed{\text{function}}$

1. When generalizing C to C', $C' \succ C$, all the examples covered by C will also be covered by C' (since if $\mathcal{B} \cup \{C\} \models e$ (e being an example) holds then also $\mathcal{B} \cup \{C'\} \models e$ holds). This property is used to prune the search of more general clauses when e **is a negative example: if e is inconsistent (covers a negative example) then all its generalizations will also be inconsistent**. Hence, the generalizations of C do not need to be considered.

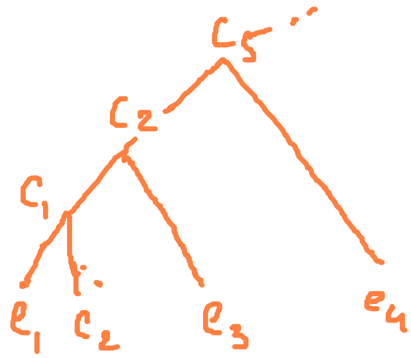
$f_{e^-}(C) \uparrow$

2. When specializing C to C', $C \succ C'$, an example not covered by C will not be covered by any of its specializations either (since if $\mathcal{B} \cup \{C\} \not\models e$ holds then also $\mathcal{B} \cup \{C'\} \not\models e$ holds). This property is used to prune the search of more specific clauses when e **is an uncovered positive example: if C does not cover a positive example none of its specializations will**. Hence, the specializations of C do not need to be considered.

$-f_{e^+}(C) \downarrow$

③ Basis for 2 ILP techniques.

① Bottom up building of LGGs from training examples, relative to background knowledge



May want some pre/post processing

② Top-down search using refinement operators (along "covers" edge)

Comparing generality ordering

$$\text{if } C \succeq_\theta D \text{ then } C \succeq_\models D$$

$$\text{but}$$

$$\text{not vice} - \text{versa}$$

For example, the above holds for:

$C:\ natural(s(X)) \leftarrow natural(X)$

$D:\ natural(s(s(X))) \leftarrow natural(X)$

Subsumption theorem explains the asymmetry

If Σ is a set of clauses and D is a clause, then $\Sigma \models D$ iff D is a tautology, or there exists a clause $D' \succeq_\theta D$ which can be derived from Σ using some form of resolution.

Σ
T_*
D'
θ ✓
D

T

$\underbrace{\text{Asymmetry}}$ is because $C \in \Sigma$ may be self-recursive OR D may be a tautology.

Principled approach to generality ordering

Given a set of clauses S, clauses $C, D \in S$ and quasi-orders $\succeq_1$ and $\succeq_2$ on S, then $\succeq_1$ is *stronger* than $\succeq_2$ if $C \succeq_2 D$ implies $C \succeq_1 D$. If also for some $C, D \in S$ $C \not\succeq_2 D$ and $C \succeq_1 D$ then $\succeq_1$ is *strictly stronger* than $\succeq_2$

$\succeq_\models$ is strictly stronger than $\succeq_\theta$

SOME GENERALITY ORDERINGS

decreasing generality

- $C \succeq_\models D$ iff $C \models D$
- $C \succeq_\theta$ iff there is a substitution θ s.t. $C \subseteq D$
- $C \succeq_{\theta'} D$ iff every literal in D is *compatible* to a literal in C and $C \succeq_\theta D$
- $C \succeq_{\theta''} D$ iff $|C| \geq |D|$ and $C \succeq_{\theta'} D$

To avoid silly cases such as $\{P(x,y)\} \succeq_\theta \{P(x,y), P(y,x)\}$

To avoid silly cases of $\succeq_\theta$ such as $\underbrace{\{P(x,y)\}}_{C} \succeq \underbrace{\{P(x',y'), Q(y')\}}_{D}$

Which generality to choose?

↳ Strongest ordering (so that you dont overgeneralize or overspecialize)

↳ But practical. (LGI is too expensive)

↳ $\models$ is undecidable

NP-complete

- Determinate Horn clauses. There exists an ordering of literals in C and exactly one substitution θ s.t. $C\theta \subseteq D$.

- $k-local$ Horn clauses. Partition a Horn clause into k "disjoint" sub-parts and perform k independent subsumption tests.

⋮

(as before def.

Subsumption between clauses is a decidable relation, whereas implication is not. The flip side is that subsumption is a weaker relation.

2. Equivalence classes under subsumption can be represented by a single reduced clause. Reduction can be undone by inverse reduction (*c.f.* Section 2.3).

3. Every finite set of clauses (function free or not)has a least generalization (LGS) and greatest specialization (GSS) under subsumption in $\mathcal{C}$. Hence $< \mathcal{C}, \succeq >$ is a lattice. The same is not true for the implication quasi-ordering $\succeq_{\models}$ (for restricted languages *lubs* for $\succeq_{\models}$ may well exist).

Order	*lub*	*glb*
$\succeq_{\theta}$	$\surd$	$\surd$
$\succeq_{\models}$	$\times$	$\surd$

4. Every finite set of Horn clauses has a least generalization (LGS) and greatest specialization (GSS) under subsumption in $\mathcal{H}$. Hence $< \mathcal{H}, \succeq >$ is a lattice. The same does not hold for the implication order.

5. The negative results for covers hold for subsumption as well as implication.

Implication story for $\mathcal{H}$ is sad

Incorporating Background knowledge :

Why $\mathcal{B}$?

$H_1 = CP(x) \leftarrow F(x)$

LGS

$D_1 = CP(x) \leftarrow S(x), F(x), D(x),$

$D_2 = CP(x) \leftarrow F(x), C(x)$

Say D_1 & D_2 are +ve examples

Say we also have $\mathcal{B}$.

$B_1 = P(x) \leftarrow C(x)$ . . D_2

$B_2 = P(x) \leftarrow D(x)$ . . . D_1

$B_3 = S(x) \leftarrow C(x)$ . . D_2

H_1

more satisfactory (TB is not a EP)

$H_B = CP(x) \leftarrow S(x), F(x), P(x)$

D_1 D_2

Note:- $H_B \cup \mathcal{B} \models D_1$ & D_2 but $H_B \not\models D_1$ & D_2

Three generality orderings with $\mathcal{B}$

① Plotkin's relative subsumption ($\succsim_{\mathcal{B}}$)

C & $\mathcal{B}$ arbit

if $\mathcal{B} = \phi$, $\succsim_{\mathcal{B}} \equiv \succsim_{\theta}$

② Relative implications ($\models_{\mathcal{B}}$)

if $\mathcal{B} = \phi$, $\models_{\mathcal{B}} \equiv \succsim_{\models}$

-ve results for covers extend.

C & $\mathcal{B}$ should be definite ← ③ Generalized subsumption ($\geq_{\mathcal{B}}$)