Lattice of Models

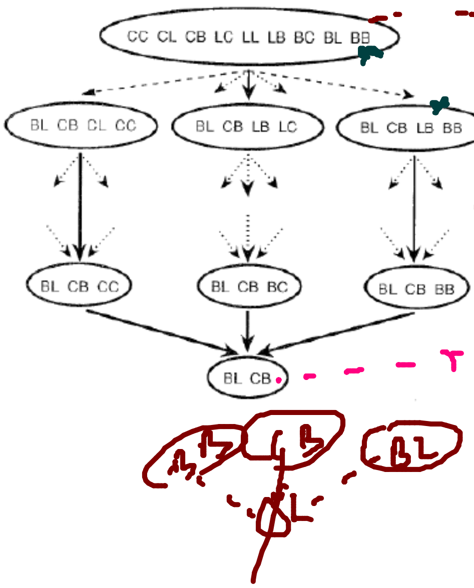
Definite clausal

$M(\Sigma)$

$M_1 \subseteq M_2$ - $M_1 \leq M_2$

$\langle M(\Sigma), \subseteq \rangle$ forms a complete lattice.

$\langle I(\Sigma), \subseteq \rangle$ forms a complet lattice



--→ Lattice of I(Σ)

Model lattice

--→ MM(Σ) ✓✓

Covers reln in Model lattice spans paths of covers in int lattice

Inductive Log Prog

(⊨)

① Learning from entailment --→ SLPs --→ FOIL --- Aleph

② Learning from interp (models) --→ MLNs --→ BLPs

③ Learning from proofs --→ Penntreebank

Claim:- If $M_1, M_2 .. M_n$ are models for def clause set Σ,

then $M_1 \cap M_2 . . . \cap M_n$ is also a model of Σ

[def cl:- 1 +ve lit, ≥ 0 -ve lit]

$A \leftarrow B, C.$ ✓ $A \vee \neg B \vee \neg C$

$B \leftarrow A, D$ ✗ $B \vee A \vee \neg D$

Proof:- Prove by induction on 'n'

$M_1 \cap M_2 .. \cap M_k = I_k$

Consider any $C \in \Sigma . \Rightarrow C : A \leftarrow B_1 .. B_m$

We will prove that $\forall k$, I_k satisfies C.

① Base case:- $k=1$

$I_1 = M_1$ - - a model

② Induction: Assume truth for $1 \leq k \leq n$

is a model for Σ

ie $I_k = M_1 \cap M_2 .. \cap M_k$

If $I_n = M_1 \cap M_2 \ldots \cap M_n \ldots (M_{n+1})$

① If I_n satisfies C, then $\rightarrow B_i \notin I_n$ for some i

$\rightarrow B_i \in I_n \;\forall i,\; A \in I_n$

Req. def clause

intersection.

If M_{n+1} satisfies C, then

$\therefore$ For $\underbrace{I_n \cap M_{n+1}}$ $\rightarrow B_i \notin I_n$ for some i

$\rightarrow B_i \in I_n \;\forall i,\; A \in I_n$

$\therefore I_{n+1}$ is also $\in \mathcal{M}(\Sigma)$

Model intersection property

② $I^* = \bigcap_{M_i \in M(\Sigma)} M_i \longrightarrow$ is minimal model $MM(\Sigma)$

$I^* = MM(\Sigma)$

$MM(\Sigma)$:- $\nexists$ no Model of Σ which $\subseteq MM(\Sigma)$

If I^* is not minimal,

$\exists M_j \in M(\Sigma)$ st $M_j \subset I^*$.

$\Rightarrow \exists q \notin M_j, q \in I^*$.

But $q \in I^* \Rightarrow q \in M_i \ \forall i$

... Contradiction.

If α is an atom then $\Sigma \models \alpha$ iff $\alpha \in MM(\Sigma)$

Theorem 3

only if : If $\Sigma \models q$, then q is true in every model of Σ

$\Rightarrow q \in I^{*}$

$\Rightarrow q \in MM(\Sigma)$

if $q \in MM(\Sigma)$, then q is true in every model of Σ

$\Rightarrow \Sigma \models q$

C_0 : CC ← CL

C_1 : CB ← ¬ BL

C_2 : CL ← LL

C_3 : BL

Procedure for enumerating MM.

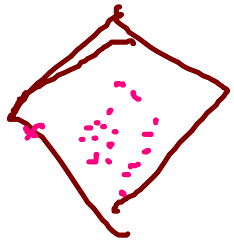
CC ← CL

CB ← BL

CL ← LL

BL ← True.

$\langle B(\Sigma), \subseteq \rangle$: Comp latt.



$I_{k+1} = T_\Sigma(I_k)$

M(Σ) is a sublattice of B(Σ)

$T_\Sigma(I) = \{a : a \leftarrow body \ \& \ body \subseteq I\}$

$A_1 \leftarrow Body_1,$

$A_2 \leftarrow body_2$ }

$I = \phi.$ $T_\Sigma(\phi) = \{BL\}$

continuous | monotonic

Theorem 4 *Let $< S, \preceq >$ be a complete lattice and let $f : S \to S$ be a monotonic function. Then the set of fixed points of f in L is also a complete lattice $< P, \preceq >$ (which obviously means that f has a greatest as well as a least fixpoint).*

Claim:- $T_\Sigma(I^\Delta) = I^\Delta$

Kleen's recursion thm.

T_Σ is cts & monotone .. iterative application of T_Σ, starting with $\perp$ el. → yield glb of the fixpoint lattice.

Kleen's Rc

$$\underbrace{I_0 = \phi}_{\perp} \longrightarrow T_\Sigma(I_0) \xrightarrow[I_1]{} T_\Sigma(I_1) \longleftrightarrow \cdots T_\Sigma(I^\Delta) \longrightarrow \underbrace{I^\Delta}_{}.$$

Least fixpt of $T_\Sigma = MM(\Sigma)$

least = glb of $M(\Sigma)$

$I^\Delta = MM(\Sigma)$

Claim: $MM(\Sigma)$ is the least fixpt of T_Σ.

Proof: Defn of $\boxed{T_\Sigma}$

$\forall I \subseteq B(\Sigma)$ & $\forall q$, $q \in T_\Sigma(I)$ iff

$(\exists\, body)[q \leftarrow body] \in \Sigma$

& $body \in I$

whereas defn of model requires

$\forall I \subseteq B(\Sigma)$, I is a model for Σ iff $\forall q$,

$q \in I$ $\boxed{\text{if}}$ $(\exists\, body)[q \leftarrow body] \in \Sigma$

& $body \subseteq I$

Jointly, $\forall I \subseteq B(\Sigma)$, I is a model for Σ iff

$\forall q$, $q \in I$ if $q \in T_\Sigma(I)$ iff

$T_\Sigma(I) \subseteq I$

Proof Sketch:[2] Let $D = \{x | x \preceq f(x)\}$. From the very definition of D, it follows that every fixpoint is in D. Consider some $x \in D$. Then because f is monotone we have $f(x) \preceq f(f(x))$. Thus,

$$\forall\ x \in D,\ f(x) \in D \tag{1.1}$$

Let $u = lub(D)$ (which should exist according to our assumption that $< S, \preceq >$ is a complete lattice. Then $x \preceq u$ and $f(x) \preceq f(u)$, so $x \preceq f(x) \preceq f(u)$. Therefore $f(u)$ is an upper bound of D. However, u is the least upper bound, hence $u \preceq f(u)$, which in turn implies that, $u \in D$. From (1.1), it follows that $f(u) \in D$. From $u = lub(D)$, $f(u) \in D$ and $u \preceq f(u)$, it follows that $f(u) = u$. Because every fixpoint is in D we have that u is the greatest fixpoint of f. Similarly, it can be proved that if $E = \{x | f(x) \preceq x\}$, then $v = glb(E)$ is a fixed point and therefore the smallest fixpoint of f. □

Kleene's First Recursion Theorem tells us how to find the element $s \in S$ that is the least fixpoint, by incrementally constructing lubs starting from applying a continuous function to the least element of the lattice ($\perp$).

$E = \{x \mid f(x) \preceq x\}$

I is a model for Σ

iff $T_\Sigma(I) \subseteq I$

$T_\Sigma(I) \preceq I$

$glb(E) = v$ → fix point

$glb(E) =$ least fix pt of T_Σ

We already proved $glb(M(\Sigma))$

$= MM(\Sigma)$

Proc:

$I_0 := \emptyset = \bot$

$I_1 := T_\Sigma(I_0)$

$I_2 := T_\Sigma(I_1)$

$\vdots$

$I^* := T_\Sigma(I^*)$

Default inference under closed world assumption

Σ_L :→ determines all knowledge to be in L

'A' :- if $\Sigma_L \models A$ then A is true else false.

default inference :- Necessary supplement to deductive inference

$\Sigma \models \alpha$ $\qquad$ $\Sigma \cup \{\neg\alpha\} \underset{\vdash}{\models} \square$

$\Sigma = \{A \leftarrow B\}$ $\qquad$ $\mathcal{B}(\Sigma) = \{A, B\}$

$MM(\Sigma) = \phi$: all atoms P, $\Sigma \models P$

} Infer ¬A since $A \notin MM(\Sigma)$ $A \in \mathcal{B}(\Sigma)$

Infer ¬A in default of Σ implying A.

Q: Soundness & completeness.

2 constructions that provide consequence oriented meaning for def inf.

(CF) [COM]   (resolution) [PROC]

COM: ① $CWA(\Sigma)$: Combination of Σ + all default inferences

$\neg A$

$\Sigma \not\models A$

$$CWA(\Sigma) = \Sigma \cup \{\neg A \mid A \in \underset{\text{atoms}}{\underline{\mathcal{B}(\Sigma)}} \ \& \ \Sigma \not\models A\}$$

$$= \Sigma \cup \{\neg A \mid A \notin MM(\Sigma)\}. \quad \neg\Sigma$$

for form

α

$\Sigma \not\models \alpha$

$\Sigma \not\models \neg\alpha$

$\neg A$

$\Sigma \models \neg A$

single query $MM(\Sigma)$

Soundness of "$\vdash_{CWA}$"

$\forall A \in \mathcal{B}(\Sigma)$, $CWA(\Sigma) \models \neg A$ if $\Sigma \vdash_{CWA} \neg A$

$MM(\Sigma)$

$M(\Sigma)$.

Completeness ---> only if

Problems with CWA

ⓐ No immediate way of writing $CWA(\Sigma)$

Lot of hard work ($MM(\Sigma)$ ...)

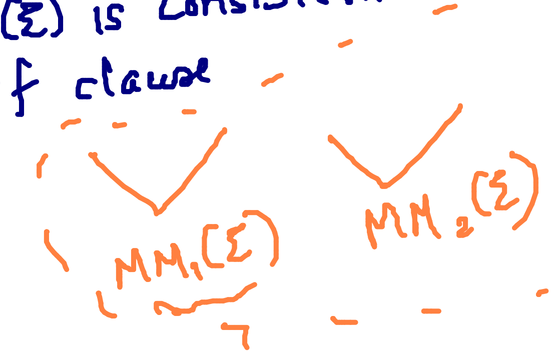
All this for a single query '$\neg A$' !

Need to infer all $B \in MM(\Sigma)$

ⓑ A more serious defect: $CWA(\Sigma)$ is consistent for def clause

$MM(\Sigma)$ exists.

$$CWA(\Sigma) = \Sigma \cup \{\neg A \mid A \notin MM(\Sigma)\}$$



CWA can be inconsistent for indef Σ.

$\Sigma = \{A \vee B\}$ $CWA(\Sigma) = \{A \vee B, \neg A, \neg B\}$ ∴ contradiction

ⓑ comp(Σ) :- useful for indef prog.

completion

completed database

→ written down directly.

More or less consistent for all "well-structured progs".

$\Sigma_1 = \{A \leftarrow \neg B\}$

biased. ⇒ defn

$(\neg B)$

$A \vee B$:- $A \leftarrow \neg B$

ⓐ Initialize $comp(\Sigma) = \emptyset$

ⓑ $\Sigma = \{A \leftarrow body\}$:- canonicalize

→ body can have -ve lit.

ⓒ $\forall A$ in $B(\Sigma)$ but NOT DEFINED in Σ,

$comp(\Sigma) = comp(\Sigma) \cup \{\neg A\}$

→ does not occur as a head.

ⓓ $A \leftarrow body_1$,

$A \leftarrow body_2$

⋮

$A \leftarrow body_n$. } if

$A \vee B$ ... $A \leftarrow \neg B$

$A \vee C$ ... $A \leftarrow \neg C$.

step ⓑ

$comp(\Sigma) \cup \Sigma \cup only\,if(\Sigma)$

$comp(\Sigma) = comp(\Sigma) \cup only\,if(\Sigma)$

$comp(\Sigma) = comp(\Sigma) \cup \{A \text{ if } [\, (body_1 \vee body_2 \ldots \vee body_n)\,]\}$

$Only\text{-}if(\Sigma)$: $body_1 \leftarrow A_1$

$body_2 \leftarrow A$

Characterization of comp(Σ).

ⓐ More economical.

does not deduce.

$\Sigma = \{A \Leftarrow B\}$

implicitly ($comp(\Sigma) = \{\neg B, A \text{ iff } B\} \models \neg A$? (Yes)

explicitly → { $CWA(\Sigma) = \{\neg A, \neg B, A \Leftarrow B\}$

Note

$\Sigma \not\models \neg A \qquad \Sigma \not\models A$

ⓑ Completion is sometimes more conservative than CWA

$\Sigma_1 = \{A \leftarrow A\}$

$comp(\Sigma_1) = \{A \text{ iff } A\}$

$CWA(\Sigma_1) = \{\neg A\}$

$comp(\Sigma_2) = \{\neg B, A \overset{if}{\cancel{iff}} \neg B\}$

$\Sigma_2 = \{A \leftarrow \neg B\}$

$\{A\}, \{B\}$.

$CWA(\Sigma_2) = \{\neg A, \neg B, A \leftarrow \neg B\}$

inconsistent

ⓒ Example comp can be less conservative

$\Sigma_3 = \{A \leftarrow \neg A\}$

$CWA(\Sigma_3) = \{A \leftarrow \neg A\} \models A \quad \not\models \neg A$

$comp(\Sigma_3) = \{A \text{ iff } \neg A\} \models$ everything.

contra.

ⓓ comp captures programmer's intentions more fully than Σ.

$A \leftarrow body_1$

$A \leftarrow body_2$

$\vdots$

ⓔ Atomic consequences of comp(Σ) could be a result of joint contns from Σ & only if (Σ) for indef clauses

For def clause :- ⓒ

ⓕ <u>syntactic bloo:-</u> (indef grammar)

$\Sigma_1 = \{\underline{A} \leftarrow \neg B\} \equiv \Sigma_2 = \{\underline{B} \leftarrow \neg A\}$

$comp(\Sigma_1) = \{\neg B, A \leftrightarrow iff \neg B\} \not\equiv comp(\Sigma_2) = \{\neg A, B \leftrightarrow \cancel{iff} \neg A\}$

$\prod$

$A, \neg B$

$\prod$

$\neg A, B.$