Attempts at ILP formulations

Let

1. $\mathcal{S}$ be the set of all spots
2. Γ be the set of all attachments
3. $\Gamma_s \subseteq \Gamma$ be the set of all attachments associated with spot $s \in \mathcal{S}$. Assuming that every $\gamma \in \Gamma$ is associated with a unique spot s, we should have that Γ_s' are disjoint, that is, $\Gamma_s \cap \Gamma_{s'} = \emptyset, \ \forall \ s, s' \in \mathcal{S}$.
4. Let $\mathcal{C} \subseteq 2^\Gamma$ be the set of (possibly overlapping) clusters. For example, these clusters could be the ones that the hierarchical agglomerative algorithm induces on Γ. For any $c \in \mathcal{C}$, $c \subseteq \Gamma$.

First attempt at ILP formulation

We define the following decision variables:

1. $z_{s\gamma} \in \{0,1\}$ for each spot s and each $\gamma \in \Gamma_s$. There are $|\Gamma|$ decision variables of this kind.

The objective is

$$\begin{aligned} \text{maximize} \quad & \sum_{s,s' \in \mathcal{S},\ s \neq s'} \left(\sum_{\gamma \in \Gamma_s} \left(\sum_{\gamma' \in \Gamma_{s'}} [z_{s\gamma} z_{s'\gamma'} R(g(\gamma)) R(g(\gamma'))] \right) \right) + \sum_{s \in \mathcal{S}} \sum_{\gamma \in \Gamma_s} [z_{s\gamma} R(f_s(\gamma))] \\ \text{subject to} \quad & 0 \leq \sum_{\gamma \in \Gamma_s} z_{s\gamma} \leq 1 \text{ for each } s \in \mathcal{S} \\ & z_{s\gamma} \in \{0,1\} \text{ for each } s \in \mathcal{S} \text{ and } \gamma \in \Gamma_s \end{aligned} \tag{1}$$

This formulation has $|\Gamma|$ decision variables and $|\mathcal{S}|$ inequality constraints.

Unfortunately, this is nonlinear integer programming formulation. In fact, it is a quadratic 0-1 programming problem[1] which has been extensively studied over the last thirty years. The second component of the objective as also the constraints are additionally separable; had it not been for the non-separability of first component of the objective, we would have at least had a separable nonlinear integer programming formulation.

Second attempt at ILP formulation

Fortunately, a non-linear term $b_{\gamma\gamma'} = z_{s\gamma} z_{s'\gamma'}$ can be linearized as follows, by introducing a new decision variable $b_{\gamma\gamma'}$:

$$\begin{aligned} 0 &\leq -2 \times b_{\gamma\gamma'} + z_{s\gamma} + z_{s'\gamma'} \\ 0 &\leq b_{\gamma\gamma'} - z_{s\gamma} - z_{s'\gamma'} + 1 \end{aligned} \tag{2}$$

This leads to the following integer linear programming formulation equivalent to (1):

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{\gamma,\gamma' \in \Gamma,\ \gamma \neq \gamma'} [b_{\gamma\gamma'} R(g(\gamma)) R(g(\gamma'))] + \sum_{s \in \mathcal{S}} \sum_{\gamma \in \Gamma_s} [z_{s\gamma} R(f_s(\gamma))] \\ \text{subject to} & 0 \leq \displaystyle\sum_{\gamma \in \Gamma_s} z_{s\gamma} \leq 1 \text{ for each } s \in \mathcal{S} \\ & 0 \leq -2 \times b_{\gamma\gamma'} + z_{s\gamma} + z_{s'\gamma'} \text{ for each } s, s' \in \mathcal{S},\ \gamma \in \Gamma_s \text{ and } \gamma' \in \Gamma_{s'} \\ & 0 \leq b_{\gamma\gamma'} - z_{s\gamma} - z_{s'\gamma'} + 1 \text{ for each } s, s' \in \mathcal{S},\ \gamma \in \Gamma_s \text{ and } \gamma' \in \Gamma_{s'} \\ & z_{s\gamma} \in \{0,1\} \text{ for each } s \in \mathcal{S} \text{ and } \gamma \in \Gamma_s \\ & b_{\gamma\gamma'} \in \{0,1\} \text{ for each } \gamma, \gamma' \in \Gamma \end{array} \tag{3}$$

This formulation has $\frac{1}{2}\left(|\Gamma|^2 + |\Gamma|\right)$ decision variables and $|\mathcal{S}| + \frac{1}{2}\left(|\Gamma|^2 - |\Gamma|\right)$ inequality constraints.

[1] See *Nonlinear Integer Programming*, `http://www.springer.com/math/book/978-0-387-29503-9`