Quadratic Optimization: Primal Active-Set Algorithm

Consider the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} + \beta \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} \end{array} \tag{1}$$

where $Q \succ 0$.

Below, we reproduce the primal active-set method (the motivation for each step was discussed in class) for optimization.

Step 1
Input a feasible point, $\mathbf{x}^0$, identify the active set $\mathcal{I}^0$, form matrix $A_{\mathcal{I}^0}$, and set $k = 0$.
Step 2
Compute $\mathbf{g}^k = Q\mathbf{x}^k + \mathbf{c}$.
Check the rank condition $rank[A_{\mathcal{I}^k}^T \;\; \mathbf{g}^k] = rank[A_{\mathcal{I}^k}^T]$. If it does not hold, go to **Step 4**.
Step 3
Solve the system $A_{\mathcal{I}^k}^T \widehat{\lambda} = \mathbf{g}^k$. If $\widehat{\lambda} \geq \mathbf{0}$, output $\mathbf{x}^k$ as the solution and stop; otherwise, remove the index that is associated with the most negative Lagrange multiplier (some $\widehat{\lambda}_t$) from $\mathcal{I}^k$.
Step 4
Compute the value of $\mathbf{d}^k$:

$$\begin{aligned} \mathbf{d}^k = \quad & \underset{\mathbf{d}}{\operatorname{argmin}} && \tfrac{1}{2}\mathbf{d}^T Q \mathbf{d} + (\mathbf{g}^k)^T \mathbf{d} \\ & \text{subject to} && \mathbf{a}_i^T \mathbf{d} = 0 \qquad \text{for } i \in \mathcal{I}^k \end{aligned} \tag{2}$$

Step 5
Compute α_k:

$$\alpha_k = \min \left\{ 1, \; \min_{\substack{j \notin \mathcal{I}^k \\ \mathbf{a}_j^T \mathbf{d}^k < 0}} \frac{\mathbf{a}_j^T \mathbf{x}^k - b_j}{-\mathbf{a}_j^T \mathbf{d}^k} \right\} \tag{3}$$

Set $\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_k \mathbf{d}^k$.
Step 6
If $\alpha_k < 1$, construct $\mathcal{I}^{k+1}$ by adding the index that yields the minimum value of α_k in (**??**). Otherwise, let $\mathcal{I}^{k+1} = \mathcal{I}^k$.
Step 7
Set $k = k + 1$ and repeat from **Step 2**.

Figure 1: Optimization for the quadratic problem in (**??**) using Primal Active-set Method.