Basis function expansion & Kernel: Part 1

We saw the that for $p \in [0, \infty)$, under certain conditions on K, the following can be equivalent representations

- $$f(\mathbf{x}) = \sum_{j=1}^{p} \underbrace{w_j\phi_j(\mathbf{x})}$$
  basis function(j)
- And
  $$f(\mathbf{x}) = \sum_{i=1}^{m} \alpha_i \underbrace{K(\mathbf{x}, \mathbf{x}_i)}$$
  basis function x_i $= h_i(x)$
- For what kind of regularizers, loss functions and $p \in [0, \infty)$ will these dual representations hold?[1] : Defer this question and deal with it after discussing classification

[1] Section 5.8.1 of Tibshi.

Basis function expansion & Kernel: Part 2

- We could also begin with

K need not be psd/mercer

$$f(\mathbf{x}) = \sum_{i=1}^{m} \alpha_i \underbrace{K(\mathbf{x}, \mathbf{x}_i)}$$

K is some measure of similarity (x, x_i)

and impose no constraints on K.

- *E.g.:* $K_k(x_q, x) = I(||x_q - x|| \leq ||x_{(k)} - x_0||)$ where $x_{(k)}$ is the training observation ranked k^{th} in distance from x and $I(S)$ is the indicator of the set S

deal with it for classification

- This is precisely the Nearest Neighbor Regression model
- Kernel regression and density models are other examples of such *local regression* methods[2]

You can basically use kernel regression to approximate functions, probability density

[2] Section 2.8.2 of Tibshi

Kernel weighted regression (Midsem Q4: Local linear regression)

Weights obtained using some kernel $K(.,.)$. Given a training set of points $\mathcal{D} = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_i, y_i), \ldots, (\mathbf{x}_n, y_n)\}$, we predict a regression function $f(x') = (\mathbf{w}'^\top \phi(x') + b')$ for each test (or query point) x' as follows:

$$(\mathbf{w}', b') = \underset{\mathbf{w}, b}{\text{argmin}} \sum_{i=1}^{n} K(x', x_i) \left(y_i - (\mathbf{w}^\top \phi(x_i) + b) \right)^2$$

1. If there is a closed form expression for $(\mathbf{w}', b')$ and therefore for $f(x')$ in terms of the known quantities, derive it.
2. How does this model compare with linear regression and $k-$nearest neighbor regression? What are the relative advantages and disadvantages of this model?
3. In the one dimensional case (that is when $\phi(x) \in \Re$), graphically try and interpret what this regression model would look like, say when $K(.,.)$ is the linear kernel[3].

[3] Hint: What would the regression function look like at each training data

More on Kernels after some classification

(By data set here we mean the training dataset)

1. We will delve a bit more into kernel density estimation etc after some treatment of classification

# of params dont change with data size

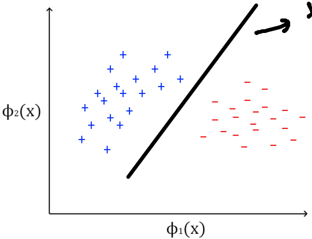
- We will talk of parametric vs non-parametric estimation

#params increase with increasing dataset size

Perceptron

& Kernel perceptron

& deep neural networks



the linear Seperator between blue (+ve) & red (-ve) points

$w^{\top}\phi(x) + b \geq 0$ *for +ve points (y= +1)*
$w^{\top}\phi(x) + b < 0$ *for -ve points (y= -1)*
$w, \phi \in \mathbb{R}^m$

in other words: $y(w^T\phi(x)+b) \geq 0 \quad \forall (x,y)$

- Assuming the problem is linearly separable, there is a learning rule that converges in a finite time.
- A new (unseen) input pattern that is similar to an old (seen) input pattern is likely to be classified correctly

- Often, b is indirectly captured by including it in w, and using a ϕ as: $\phi_{aug} = [\phi, 1]$
- Thus, $w^\top \phi(x)$

$$= \begin{bmatrix} w_1 & w_2 & w_3 & \dots & w_m & b \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_m \\ 1 \end{bmatrix}$$

- $\boxed{w^\top \phi(x) = 0 \text{ is the separating hyperplane.}}$

Perceptron Intuition

$\{(\phi(x_1), y_1), (\phi(x_2), y_2) \ldots (\phi(x_n), y_n)\}$

- Go over all the existing examples, whose class is known, and check their classification with a current weight vector $(w^{(0)})$
- If correct, continue
- If not, add to the weights a quantity that is proportional to the product of the input pattern with the desired output y $(1$ or $-1)$

Exercise: 1 Write down this perceptron (update) algorithm
2 Geometrically interpret the updates.
3 Will it converge for linearly separable dataset? Why?

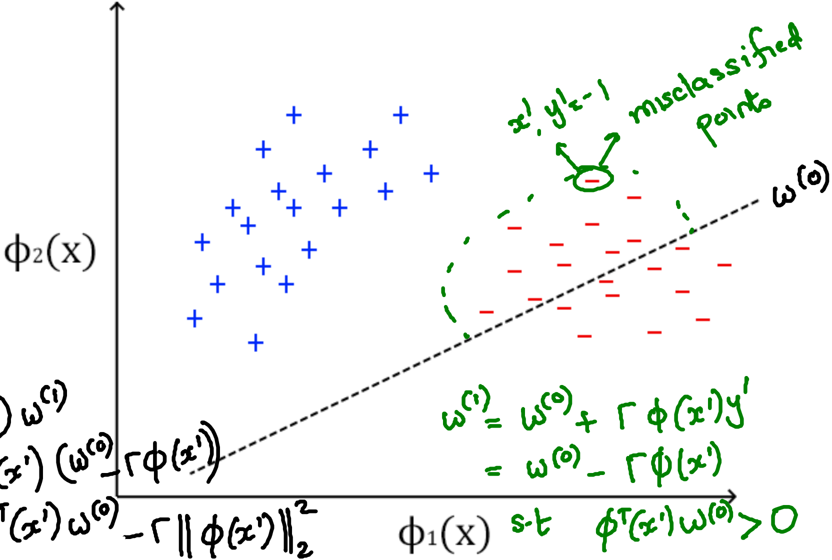
Perceptron Update Rule

$w^{(0)} \neq 0$

$w^{(0)} = [0\ 0 \cdots 1]$

- Start with some weight vector $w^{(0)}$, and for $k = 1, 2, 3, \ldots, n$ (for every example), do:
  $w^{(k)} = w^{(k-1)} + \Gamma\phi(x')$
- where x' s.t. x' is misclassified by $(w^{(k)})^\top\phi(x)$
  i.e. $y'(w^{(k)})^\top\phi(x') < 0$ → This step more complex for neural nets & graphical models!

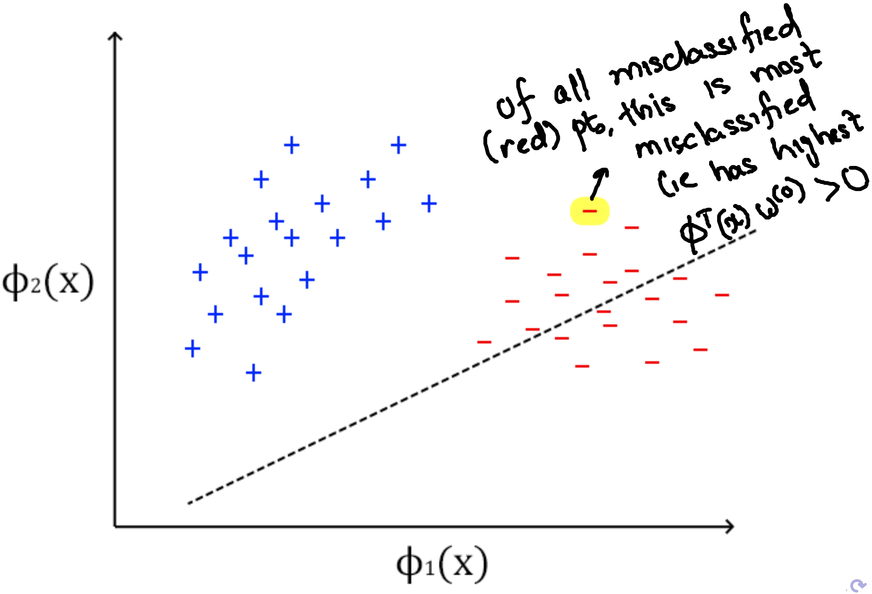
A very handy & simple algo:
1. At the core of neural network & several graphical model training
2. It is "stochastic", dealing with one point at a time & can be used for incremental training



$$w^{(1)} = w^{(0)} + \Gamma \phi(x') y' = w^{(0)} - \Gamma \phi(x')$$

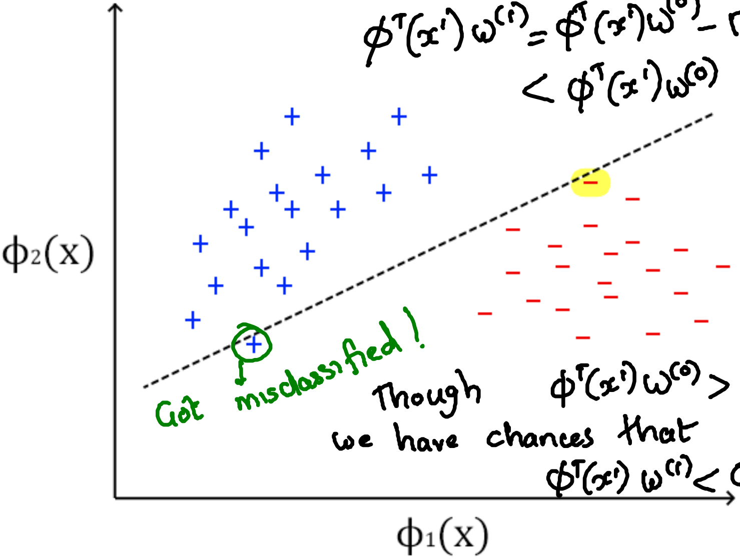
s.t. $\phi^T(x') w^{(0)} > 0$

$$\phi^T(x') w^{(1)} = \phi^T(x')\left(w^{(0)} - \Gamma \phi(x')\right) = \phi^T(x') w^{(0)} - \Gamma \|\phi(x')\|_2^2$$


Of all misclassified (red) pts, this is most misclassified (ie has highest $\phi^T(x)\, w^{(0)} > 0$
$\phi_2(x)$
$\phi_1(x)$

From what we saw:

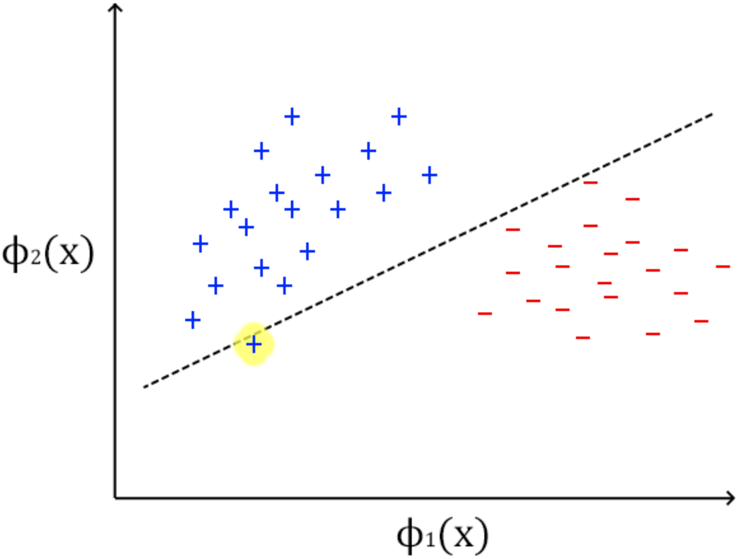
$$\phi^T(x')\,w^{(1)} = \phi^T(x')\,w^{(0)} - r\,\|\phi(x')\|_2^2 < \phi^T(x')\,w^{(0)}$$



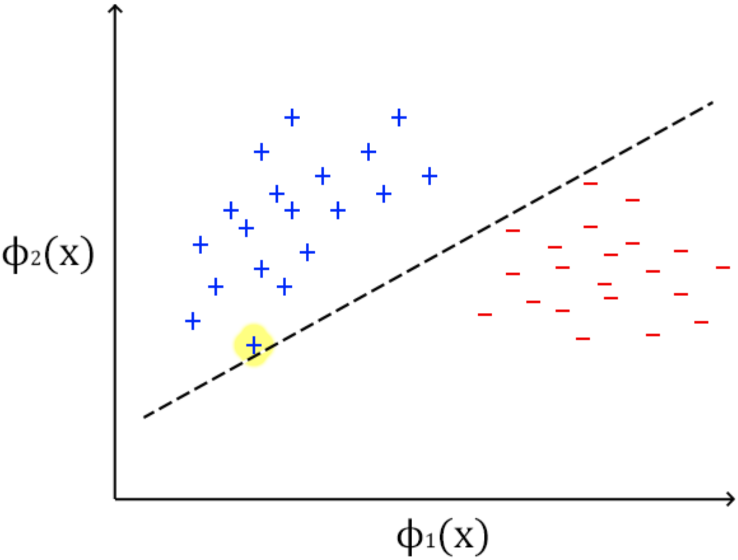
Got misclassified!

Though $\phi^T(x')\,w^{(0)} > 0$

we have chances that $\phi^T(x')\,w^{(1)} < 0$



ϕ2(x)
ϕ1(x)


ϕ2(x)
ϕ1(x)

- Perceptron does not find the *best* seperating hyperplane, it finds *any* seperating hyperplane.
- In case the initial w does not classify all the examples, the seperating hyperplane corresponding to the final w^* will often pass through an example.
- The seperating hyperplane does not provide enough breathing space – this is what SVMs address!

H/W: Go through proof of convergence of perceptron update!