Tutorial 3

Friday 29$^{\text{th}}$ January, 2016; 23:17

Problem 1. Which of the following sets are convex?

1. A slab, i.e., a set of the form $\{x \in \mathbb{R}^n | \alpha \leq a^T x \leq \beta\}$.
2. A rectangle, i.e., a set of the form $\{x \in \mathbb{R}^n | \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$.
3. A wedge, i.e., $\{x \in \mathbb{R}^n | a_1^T x \leq b_1, a_2^T x \leq b_2\}$.
4. The set of points closer to a given point than a given set, i.e., $\{x| \; ||x - x_0||_2 \leq ||x - y||_2 \text{ for all } y \in S\}$ where $S \subset \mathbb{R}^n$.
5. The set of points closer to a set than another, i.e., $\{x|\text{dist}(x, S) \leq \text{dist}(x, T)\}$, where $S, T \subset \mathbb{R}^n$, and $\text{dist}(x, S) = \inf\{||x - z||_2 | z \in S\}$.
6. The set $\{x | x + S_2 \subset S_1\}$, where $S_1, S_2 \subset \mathbb{R}^n$ with S_1 convex.
7. The set of points whose distance to a does not exceeds a fixed fraction θ of the distance to b, i.e., the set $\{x| \; ||x - a||_2 \leq \theta ||x - b||_2\}$. Assume $a \neq b$ and $0 \leq \theta \leq 1$.

Besides trying to prove a set convex, try to plot the set into two dimensions to get a feel of the shape of the set. (Exercise 2.12 of Convex Optimization, Boyd et al http://stanford.edu/ boyd/cvxbook/)

Problem 2. Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is convex, and $a, b \in \text{dom} f$ with $a < b$.

1. Show that
$$f(x) \leq \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b)$$
for all $x \in [a, b]$.
2. Show that
$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$
for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.
3. Suppose f is differentiable. Use the result in (2) to show that
$$f'(a) \leq \frac{f(b) - f(a)}{b - a} \leq f'(b)$$

4. Suppose f is twice differentiable. Use the result in (3) to show that $f''(a) \geq 0$ and $f''(b) \geq 0$.

(Exercise 3.1 of Convex Optimization, Boyd et al http://stanford.edu/ boyd/cvxbook/)

Problem 3. Determine which one is convex/concave:

1. $f(x) = e^x - 1$ on $\mathbb{R}$.
2. $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
3. $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.
4. $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.
5. $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.
6. $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}),$ where $0 \leq \alpha \leq 1,$ on $\mathbb{R}^2_{++}$.

(Exercise 3.16 of Convex Optimization, Boyd et al http://stanford.edu/ boyd/cvxbook/)

Problem 4. Suppose $p \geq 1$. Show that the function

$$f(x) = \left(\sum_{i=1}^{n} x_i^P \right)^{1/p}$$

which $\mathbf{dom} f = \mathbb{R}^n_{++}$ is convex.

Now suppose $p < 1, p \neq 0$. Show that f is concave on the same domain. (Exercise 3.17 of Convex Optimization, Boyd et al http://stanford.edu/ boyd/cvxbook/)

Problem 5. Consider a data set in which each data point y_i is associated with a weighting factor r_i, so that the sum-square error function becomes

$$\frac{1}{2} \sum_{i=1}^{m} r_i (y_i - w^T \phi(x_i))^2$$

Find an expression for the solution w^* that minimizes this error function. The weights r_i's are known before hand. (Exercise 3.3 of Pattern Recognition and Machine Learning, Christopher Bishop)