Introduction to Machine Learning - CS725
Instructor: Prof. Ganesh Ramakrishnan
Lecture 11 - Constrained Optimization, KKT Conditions, Duality, SVM Dual

Convex Optimization Problem

- Formally, a convex optimization problem is an optimization problem of the form

$$
\begin{aligned}
\textit{minimize } & f(\mathbf{x}) && (1)\\
\textit{subject to } c \ \in & \ C && (2)
\end{aligned}
$$

  where f is a convex function, C is a convex set, and **x** is the optimization variable.

- A specific form of the above would be

$$
\begin{aligned}
\textit{minimize } f(\mathbf{x}) & && (3)\\
\textit{subject to } g_i(\mathbf{x}) & \leq 0, \ i = 1, ..., m && (4)\\
h_i(\mathbf{x}) & = 0, i = 1, ..., p && (5)
\end{aligned}
$$

  where f is a convex function, g_i are convex functions, and h_i are affine (linear) functions, and **x** is the vector of optimization variables.

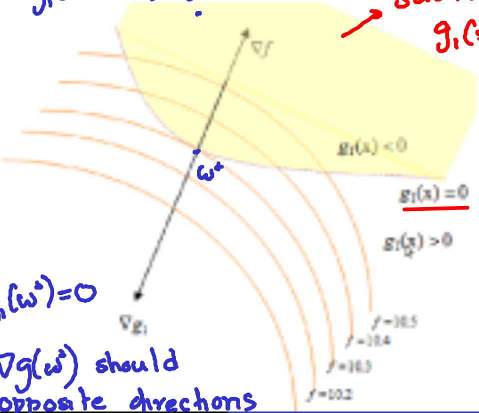
Q. *How to solve constrained problems of the above-mentioned type?*

A. Canonical example:

$$\text{Minimize } f(\mathbf{w}) \text{ s.t. } g_1(\mathbf{w}) \leq 0 \tag{6}$$

(a) [Case of constraint being naturally followed] → if $g_1(w^*) < 0 \Rightarrow \nabla f(w^*) = 0$

sublevel set of $g_1(x)$ at 0 $= \{x \mid g_1(x) \leq 0\}$



(b) [When constraint enforcement is effective] → if $g_1(w^*) = 0$

$\Rightarrow \nabla f(w^*)$ & $\nabla g(w^*)$ should point in opposite directions

Constrained Convex Problems

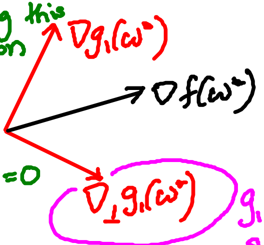
- If $\mathbf{w}^*$ is on the boundary of g_1, *i.e.*, if $g_1(\mathbf{w}^*) = 0$,

$$\nabla f(\mathbf{w}^*) = -\lambda \nabla g_1(\mathbf{w}^*) \text{ for some } \lambda \geq 0$$

- **Intuition:**

$$\nabla f(w^*) = -\lambda_1 \nabla g_1(w^*) + \lambda_2 \nabla_{\perp} g_1(w^*)$$

g_1 increases along this direction



① $\lambda_2 = 0$

② $\lambda_1 \geq 0$

Else: descent along f & g_1 simultaneously!

If w^* is pt of optimality at $g_1(w^*) = 0$ then decreasing f & g_1 simultaneously should be impossible!

g_1 will not change. As such we are at w^* s.t $g_1(w^*) = 0$

[1] $\nabla_{\perp} g_1(\mathbf{w}^*)$ is the direction orthogonal to $\nabla g_1(\mathbf{w}^*)$

[2] Section 4.4, pg-72: cs725/notes/BasicsOfConvexOptimization.pdf

Constrained Convex Problems

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$$\nabla f(\mathbf{w}^*) = -\lambda \nabla g_1(\mathbf{w}^*) \textit{ for some } \lambda \geq 0$$

- **Intuition:** If the above didn't hold, then we would have $\nabla f(\mathbf{w}^*) = \lambda_1 \nabla g_1(\mathbf{w}^*) + \lambda_2 \nabla_\perp g_1(\mathbf{w}^*)$, where, by moving in direction[1] $\pm \nabla_\perp g_1(\mathbf{w}^*)$ (or $-\nabla g_1(\mathbf{w}^*)$), we remain on boundary $g_1(\mathbf{w}^*) = 0$, (or within $g_1(\mathbf{w}^*) \leq 0$) while decreasing the value of f, which is not possible at the point of optimality.
- Thus, at the point of optimality[2],

$$\nabla f(w^*) + \lambda \nabla g_1(w^*) = 0 \quad \text{if } g_1(w^*) = 0$$

$$\nabla f(w^*) = 0 \quad \text{if } g_1(w^*) < 0$$

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Constrained Convex Problems

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- Thus, at the point of optimality[2], for some $\lambda \geq 0$,

Combine by $\lambda g(\omega^*) = 0$

$$\textit{Either } g_1(\mathbf{w}^*) < 0 \quad \& \quad \nabla f(\mathbf{w}^*) = 0 \qquad (7)$$

$$\textit{Or } g_1(\mathbf{w}^*) = 0 \quad \& \quad \nabla f(\mathbf{w}^*) = -\lambda \nabla g_1(\mathbf{w}^*) \qquad (8)$$

$L(\omega,\lambda) = f(\omega) + \lambda g_1(\omega)$

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Explaining the Figure

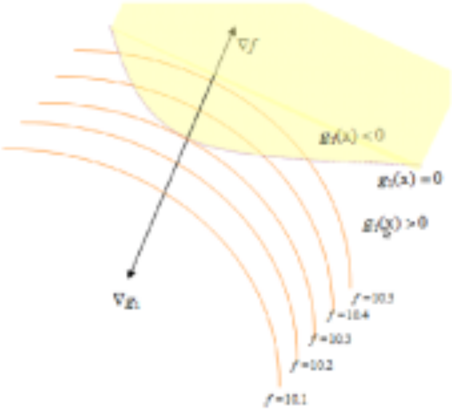


Figure 2: Two conditions under which a minimum can occur: a) When the minimum is on the constraint function boundary, in which case the gradients are in opposite directions; b) When point of minimum is inside the constraint space (shown in yellow shade), in which case $\nabla f(\mathbf{w}^*) = \mathbf{0}$.

More Explanation and Lagrange Function

- The first condition occurs when minima lies on the boundary of function g. In this case, gradient vectors corresponding to the functions f and g, at $\mathbf{w}^*$, point in opposite directions barring multiplication by a real constant.
- Second condition represents the case that point of minimum lies inside the constraint space. This space is shown shaded in Figure 1. Clearly, for this case, $\nabla f(\mathbf{w}) = \mathbf{0}$.
- An Alternative Representation: $\nabla L(\mathbf{w}, \lambda) = 0$ for some $\lambda \geq 0$ where

$$L(\mathbf{w}, \lambda) = f(\mathbf{w}) + \lambda \mathbf{g}(\mathbf{w}); \lambda \in \mathbb{R}$$

is called the lagrange function which has objective function augmented by weighted sum of constraint functions

$\{w \mid h(w)=0\} = \{w \mid h(w) \leq 0, -h(w) \leq 0\}$ .. For special case of equality constraints

Duality and KKT conditions

Karush Kuhn Tucker

For a convex objective and constraint function, the minima, $\mathbf{w}^*$, can satisfy one of the following two conditions:

1. $g(\mathbf{w}^*) = \mathbf{0}$ and $\nabla f(\mathbf{w}^*) = -\lambda \nabla \mathbf{g}(\mathbf{w}^*)$
2. $g(\mathbf{w}^*) < \mathbf{0}$ and $\nabla f(\mathbf{w}^*) = \mathbf{0}$

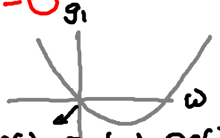
Lagrange fn Lagrange multiplier

Making use of Lagrange $L(\omega, \lambda) = f(\omega) + \lambda g(\omega)$

① $\nabla_\omega L(\omega^*, \lambda^*) = \nabla f(\omega^*) + \lambda^* \nabla g(\omega^*) = 0$

② $\lambda^* g(\omega^*) = 0$ [complementary slackness]

③ $\lambda^* \geq 0$ ④ $g(\omega^*) \leq 0$



$g(0) = 0$ but $\nabla g(0) \neq 0$

$g_1(w)$

- Here, we wish to penalize higher magnitude coefficients, hence, we wish $g(\mathbf{w})$ to be negative while minimizing the lagrangian. In order to maintain such direction, we must have $\lambda \geq 0$. Also, for solution $\mathbf{w}$ to be feasible, $g(\mathbf{w}) \leq \mathbf{0}$.
- Due to complementary slackness condition, we further have $\lambda g(\mathbf{w}) = \mathbf{0}$, which roughly suggests that the lagrange multiplier is zero unless constraint is active at the minimum point. As $\mathbf{w}$ minimizes the lagrangian $L(\mathbf{w}, \lambda)$, gradient must vanish at this point and hence we have $\nabla f(\mathbf{w}) + \lambda \nabla \mathbf{g}(\mathbf{w}) = \mathbf{0}$