Lecture 15: Kernel perceptron, Neural Networks, SVMs etc

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Perceptron Update Rule: Basic Idea

- Perceptron works for two classes ($y = \pm 1$). A point is misclassified if $y\mathbf{w}^T(\phi(\mathbf{x})) < 0$

negative unsigned distance

- Perceptron Algorithm:
  - INITIALIZE: w=ones() [for any initialization vector]
  - REPEAT: for each $< \mathbf{x}, y >$
    - ★ If $y\mathbf{w}^T\Phi(\mathbf{x}) < 0$
    - ★ then, $\mathbf{w} = \mathbf{w} + \eta\phi(\mathbf{x}).y$
    - ★ endif

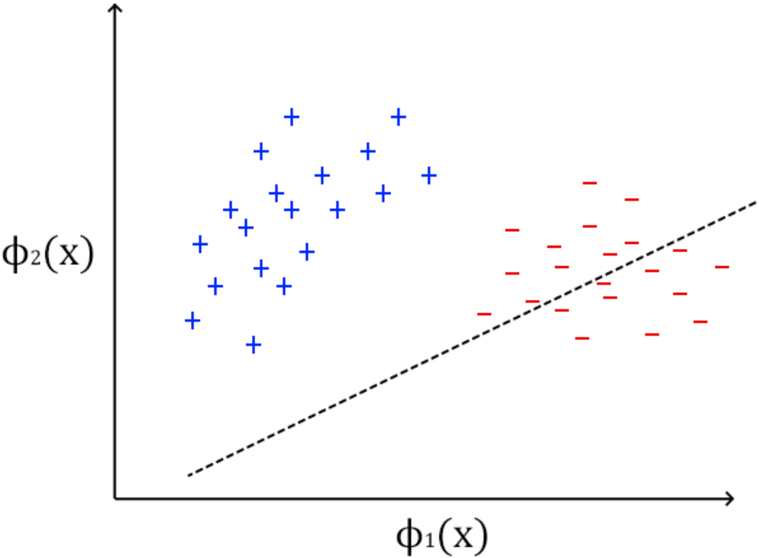
η → learning rate

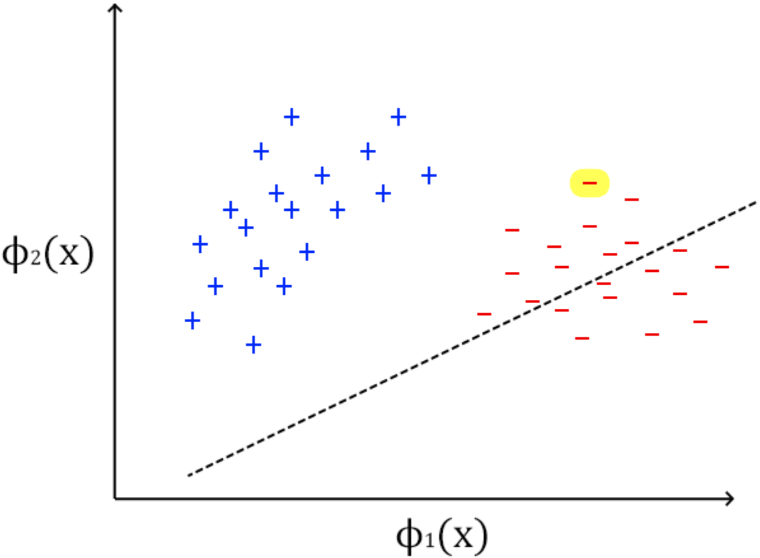
- **Intuition:**

$$
\begin{aligned}
y(\mathbf{w}^{(k+1)})^T\phi(\mathbf{x}) &= y\left(\mathbf{w}^k + \eta y\phi^T(\mathbf{x})\right)\phi(\mathbf{x}) \\
&= y(\mathbf{w}^k)^T\phi(\mathbf{x}) + \eta y^2\|\phi(\mathbf{w})\|^2 \\
&> y(\mathbf{w}^k)^T\phi(\mathbf{x})
\end{aligned}
$$

Unsigned distance (initially negative) becomes more positive)

Since $y(\mathbf{w}^k)^T\phi(\mathbf{x}) \leq 0$, we have $y(\mathbf{w}^{(k+1)})^T\phi(\mathbf{x}) > y(\mathbf{w}^k)^T\phi(\mathbf{x}) \Rightarrow$ more hope that this point is classified correctly now.


$\phi_2(x)$
$\phi_1(x)$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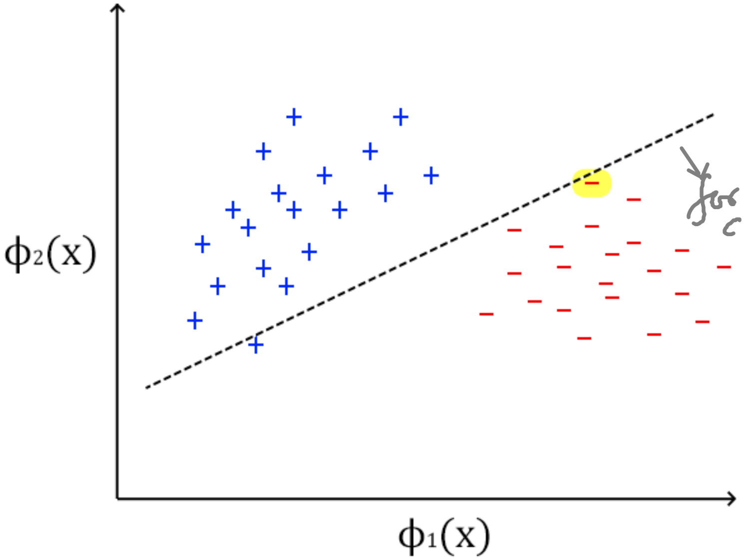
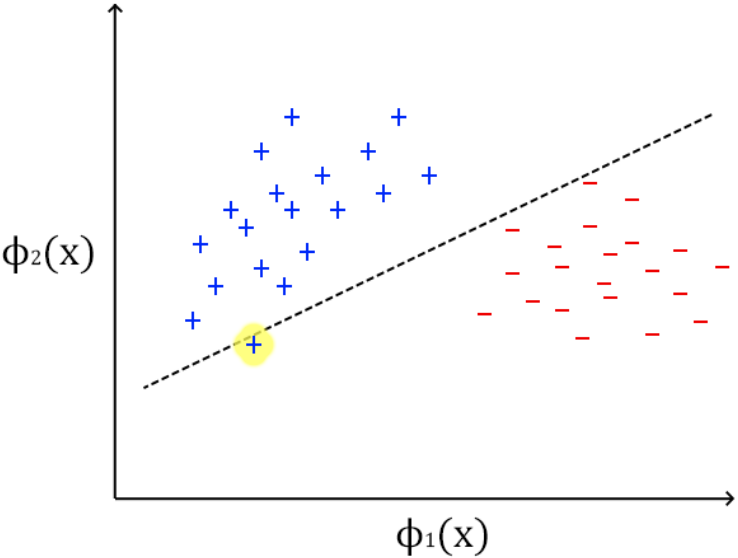


for some other choice of η this point could have yet remained (less) misclassified


$\phi_2(x)$
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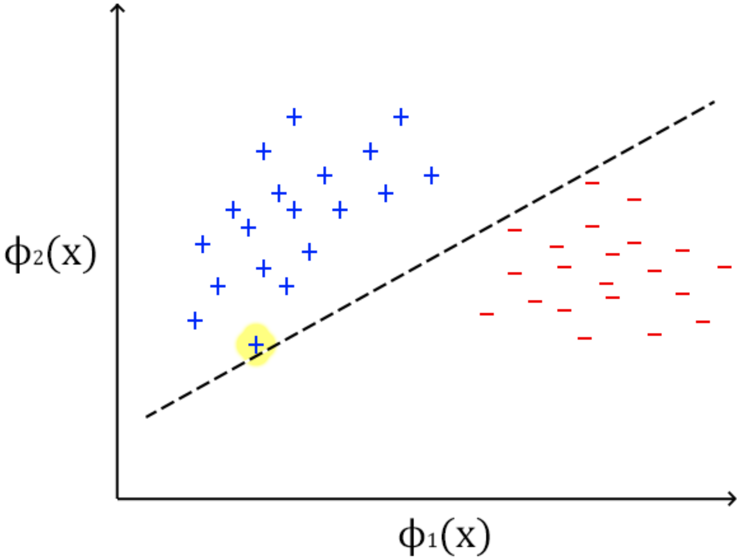
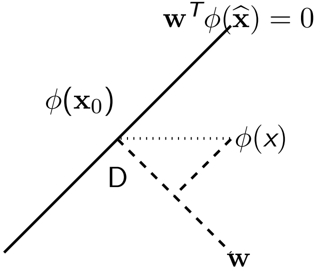


ϕ2(x)
ϕ1(x)

Perceptron Update Rule: Error Perspective

- Explicitly account for signed distance of (misclassified) points from the hyperplane $\mathbf{w}^T\phi(\widehat{\mathbf{x}}) = 0$. Consider point $\mathbf{x}_0$ such that $\mathbf{w}^T(\phi(\mathbf{x}_0)) = 0$
- (Signed) Distance from hyperplane is: $\mathbf{w}^T(\phi(\mathbf{x}) - \phi(\mathbf{x}_0)) = \mathbf{w}^T(\phi(\mathbf{x}))$
- Unsigned distance from hyperplane is: $y\mathbf{w}^T(\phi(\mathbf{x}))$ (assumes correct classification)



$\text{sign}(y) = -\text{sign}(w^T\phi(x))$

≥ 0

- If $\mathbf{x}$ is misclassified, the misclassification cost for $\mathbf{x}$ is $-y\mathbf{w}^T(\phi(\mathbf{x}))$

Perceptron Update Rule: Error Minimization

- Perceptron update tries to minimize the error function E = negative of sum of unsigned distances over misclassified examples = sum of misclassification costs

$$E = -\sum_{(\mathbf{x},y)\in\mathcal{M}} y\mathbf{w}^T\phi(\mathbf{x})$$

where $\mathcal{M} \subseteq \mathcal{D}$ is the set of misclassified examples.

- **Gradient Descent (Batch Perceptron) Algorithm** $\nabla_{\mathbf{w}}E = -\sum_{(\mathbf{x},y)\in M} y\phi(\mathbf{x})$

to minimize E

$$\begin{aligned}\mathbf{w}^{(k+1)} &= \mathbf{w}^k - \eta\nabla_{\mathbf{w}}E \\ &= \mathbf{w}^k + \eta\sum_{(\mathbf{x},y)\in\mathcal{M}} y\phi(\mathbf{x})\end{aligned}$$

Perceptron Update Rule: Error Minimization

- Batch update considers all misclassified points simultaneously

$$\left.\begin{aligned} \mathbf{w}^{(k+1)} &= \mathbf{w}^k - \eta \nabla_{\mathbf{w}} E \\ &= \mathbf{w}^k + \eta \sum_{(\mathbf{x},y)\in\mathcal{M}} y\phi(\mathbf{x}) \end{aligned}\right\} \quad E = \sum_{(x,y)\in\mathcal{M}} E(x)$$

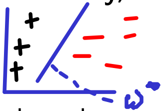
- Perceptron update $\Rightarrow$ *Stochastic Gradient Descent*:

$$\nabla_{\mathbf{w}} E = -\sum_{(\mathbf{x},y)\in\mathcal{M}} y\phi(\mathbf{x}) = -\sum_{(\mathbf{x},y)\in\mathcal{M}} \nabla_{\mathbf{w}} E(\mathbf{x}) \text{ s.t. } E(\mathbf{x}) = -y\mathbf{w}^T\phi(\mathbf{x})$$

$$\begin{aligned} \mathbf{w}^{(k+1)} &= \mathbf{w}^k - \eta \nabla_{w} E(\mathbf{x}) \qquad \text{(for any } (\mathbf{x}, y) \in \mathcal{M}) \\ &= \mathbf{w}^k + \eta y \phi(\mathbf{x}) \end{aligned}$$

Perceptron Update Rule: Further analysis

- **Formally,**:- If $\exists$ an optimal separating hyperplane with parameters $\mathbf{w}^*$ such that,



$$\forall\ (\mathbf{x}, y),\ \ y\phi^T(\mathbf{x})\mathbf{w}^* \geq 0$$

w.r.t w^*, unsigned distances are all absolute values

then the perceptron algorithm converges.

Proof:- We want to show that

$$\lim_{k\to\infty} \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 = 0 \tag{1}$$

(If this happens for some constant ρ, we are fine.)

$\|w^{(k+1)} - \rho w^*\|_2^2 = \|w^k + y\phi(x^k) - \rho w^*\|_2^2$

$= \|w^k - \rho w^*\|_2^2 + \eta y\phi^T(x^k)(w^k - \rho w^*) + y^2\|\phi(x^k)\|_2^2\eta^2$

$\leq -\theta^2 \Rightarrow \|w^{(0)} - \rho w^*\| / \theta^2 \approx$ # of iterations for convergence

We are happy with the existence of such a $\rho > 0$

θ should be independent of iterate # k

Perceptron Update Rule: Further analysis

- **Formally,**:- If $\exists$ an optimal separating hyperplane with parameters $\mathbf{w}^*$ such that,

$$\forall\ (\mathbf{x}, y),\ \ y\phi^T(\mathbf{x})\mathbf{w}^* \geq 0$$

  then the perceptron algorithm converges.
  Proof:- We want to show that

$$\lim_{k\to\infty} \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 = 0 \tag{1}$$

(If this happens for some constant ρ, we are fine.)

$$\|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 = \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 + \underline{\|y\phi(\mathbf{x})\|^2 + 2y(\mathbf{w}^k - \rho\mathbf{w}^*)^T\phi(\mathbf{x})} \tag{2}$$

- For convergence of perceptron, we need L.H.S. to be less than R.H.S. at every step, although by some small but non-zero value (with $\theta \neq 0$)

$$\|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 \leq \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 \underline{- \theta^2} \tag{3}$$

Perceptron Update Rule: Further analysis

- Need that $\|\mathbf{w}^{(k+1)} - \rho \mathbf{w}^*\|^2$ reduces by atleast θ^2 at every iteration.

$$\|\mathbf{w}^{(k+1)} - \rho \mathbf{w}^*\|^2 \leq \|\mathbf{w}^{k} - \rho \mathbf{w}^*\|^2 - \theta^2 \quad (4)$$

- Based on (2) and (4), we need to find θ such that,

Perceptron Update Rule: Further analysis

- Need that $\|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2$ reduces by atleast θ^2 at every iteration.

$$\|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 \leq \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 - \theta^2 \tag{4}$$

- Based on (2) and (4), we need to find θ such that,

$$\|\phi(\mathbf{x})\|^2 + 2y(\mathbf{w}^k - \rho\mathbf{w}^*)^T\phi(\mathbf{x}) \leq -\theta^2$$

  ($\|y\phi(\mathbf{x})\|^2 = \|\phi(\mathbf{x})\|^2$ since $y = \pm 1$)

- The number of iterations would be: $O\left(\frac{\|\mathbf{w}^{(0)} - \rho\mathbf{w}^*\|^2}{\theta^2}\right)$
- Tutorial 6, Problem 4 is concerning the number of iterations. But first we will discuss how convergence holds in the first place!

Perceptron Update Rule: Further analysis

The expression of interest : $\|\phi(x)\|^2 + \underline{2y}(w^k - \underline{\rho w^*})^T \underline{\phi(x)}$

$2y(w^k)^T\phi(x) < 0$

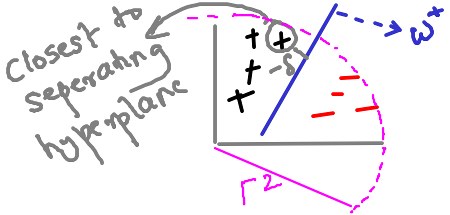
- **Observations**:-
  1. $y(\mathbf{w}^k)^T\phi(\mathbf{x}) < 0$ ($\because$ $\mathbf{x}$ was misclassified) $\leftarrow$
  2. $\Gamma^2 = \max\limits_{\mathbf{x}\in\mathcal{D}} \|\phi(\mathbf{x})\|^2 \geq \|\phi(x)\|^2$
  3. $\delta = \max\limits_{\mathbf{x}\in\mathcal{D}} -2y{\mathbf{w}^*}^T\phi(\mathbf{x}) = -\min\limits_{x\in\mathcal{D}} 2y{w^*}^T\phi(x) \Rightarrow -2y(w^*)^T\phi(x) \leq \delta$

- Here, negative margin $\delta = -2y{\mathbf{w}^*}^T\phi(\widehat{\mathbf{x}})$ is the negative of unsigned distance of closest point $\widehat{\mathbf{x}}$ from separating hyperplane : $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{D}}{\text{argmax}} -2y{\mathbf{w}^*}^T\phi(\mathbf{x}) = \underset{\mathbf{x}\in\mathcal{D}}{\text{argmin}}\, y{\mathbf{w}^*}^T\phi(x)$

- Since the data is linearly separable,

$\delta < 0$

Γ^2 = radius of tightest ball enclosing all points in $\mathcal{D}$



Perceptron Update Rule: Further analysis

Need: $\rho > 0$ st $\Gamma^2 + \rho\delta < 0$ where $\Gamma^2 > 0$ $\delta < 0$

Ans: $\rho = \frac{-2\Gamma^2}{\delta} > 0$

- **Observations**:-
  1. $y(\mathbf{w}^k)^T\phi(\mathbf{x}) < 0$ ($\because$ $\mathbf{x}$ was misclassified)
  2. $\Gamma^2 = \max_{\mathbf{x}\in\mathcal{D}} \|\phi(\mathbf{x})\|^2$
  3. $\delta = \max_{\mathbf{x}\in\mathcal{D}} -2y\mathbf{w}^{*T}\phi(\mathbf{x})$
- Here, negative margin $\delta = -2y\mathbf{w}^{*T}\phi(\widehat{\mathbf{x}})$ is the negative of unsigned distance of closest point $\widehat{\mathbf{x}}$ from separating hyperplane : $\widehat{\mathbf{x}} = \operatorname{argmax}_{\mathbf{x}\in\mathcal{D}} -2y\mathbf{w}^{*T}\phi(\mathbf{x}) = \operatorname{argmin}_{\mathbf{x}\in\mathcal{D}} y\mathbf{w}^{*T}\phi(x)$
- Since the data is linearly separable, $\widehat{y}\mathbf{w}^{*T}\phi(\widehat{\mathbf{x}}) \geq 0$, so, $\delta \leq 0$. Consequently:

$$0 \leq \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 < \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 + \Gamma^2 + \rho\delta \rightarrow -\theta^2$$

$\Gamma^2 \geq \|\phi(x)\|^2$

$\delta \geq \max_{x\in D} -y w^T\phi(x)$

Next: we show $\exists$ a $\rho \geq 0$ that gives us $\theta^2 \geq 0$

Perceptron Update Rule: Further analysis

- Since, ${\mathbf{w}^*}^T\phi(\widehat{\mathbf{x}}) \geq 0$, so, $\delta \leq 0$. Consequently:

$$0 \leq \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 < \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 + \Gamma^2 + \rho\delta$$

Taking,

Perceptron Update Rule: Further analysis

- Since, $\mathbf{w^*}^T\phi(\widehat{\mathbf{x}}) \geq 0$, so, $\delta \leq 0$. Consequently:

$$0 \leq \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 < \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 + \Gamma^2 + \rho\delta$$

Taking, $\rho = \frac{2\Gamma^2}{-\delta}$, (Any $\delta \geq \frac{2\Gamma^2}{-\delta}$ will work)

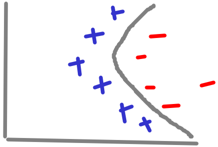
$$0 \leq \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2 \leq \|\mathbf{w}^k - \rho\mathbf{w}^*\|^2 - \Gamma^2$$

- Hence, we got, $\Gamma^2 = \theta^2$, that we were looking for in eq.(3).
  $\therefore \|\mathbf{w}^{(k+1)} - \rho\mathbf{w}^*\|^2$ decreases by atleast Γ^2 at every iteration.
- Summarily: $\mathbf{w}^k$ converges to $\rho\mathbf{w}^*$ by making a minimum θ^2 decrement at each step.
- Thus, for $k \to \infty$, $\|\mathbf{w}^k - \rho\mathbf{w}^*\| \to 0$. This proves convergence.

Perceptron Update Rule: Further analysis

- A statement on number of iterations for convergence:
  If $||\mathbf{w}^*|| = 1$ and if there exists $\delta > 0$ such that for all $i = 1, \ldots, n$, $y_i(\mathbf{w}^*)^T\phi(\mathbf{x}_i) \geq \delta$ and $||\phi(\mathbf{x}_i)||^2 \leq \Gamma^2$ then the perceptron algorithm will make atmost $\frac{\Gamma^2}{\delta^2}$ errors (that is take atmost $\frac{\Gamma^2}{\delta^2}$ iterations to converge)

Non-linear perceptron?



- Kernelized perceptron: Non linear seperation in implicit ϕ space?

Hint: Recall for non-parametric regression

$$f(x) = \sum_{i=1}^{m} \alpha_i \underbrace{K(x, x_i)}_{\phi_i(x)} y_i \rightarrow y_i \in \mathbb{R}$$

Ans: $$f(x) = \sum_{i=1}^{m} \alpha_i K(x, x_i) y_i \rightarrow y_i \in \{+1, -1\}$$

$$\begin{cases} \text{if } f(x) > 0 & y = +1 \\ f(x) < 0 & y = -1 \end{cases}$$

Non-linear perceptron?

- Kernelized perceptron: $f(\mathbf{x}) = sign\left(\sum_i \alpha_i y_i K(\mathbf{x}, \mathbf{x}_i) + b\right)$
  - INITIALIZE: α=zeroes()
  - REPEAT: for $< \mathbf{x}_i, y_i >$
    - If $sign\left(\sum_j \alpha_j y_j K(\mathbf{x}_j, \mathbf{x}_j) + b\right) \neq y_i$ → Like before check if x_i is misclassified
    - then, $\alpha_i = \alpha_i + 1$ → if x_i is indeed misclassified, increment its weight α_i by 1 !
    - endif

(Tutorial 6) is abt correspondence with perceptron

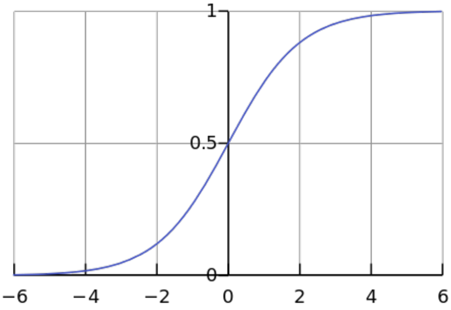
- Neural Networks: Cascade of layers of perceptrons giving you non-linearity. But before that, we will discuss the specific sigmoidal percentron used most often in Neural Networks

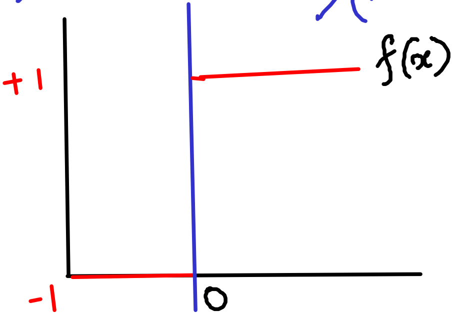
Problem with $f(x) = sign(w^T \phi(x))$ is the nondifferentiability of $f(x)$ & therefore of $E = -\sum_{(x,y)\in D} y \phi^T(x) w \lambda_w(x)$

$\lambda_w(x) = 1$ if x is misclassified

Sigmoidal (perceptron) Classifier [Note: $y \in \{0,1\}$ against $y \in \{+1,-1\}$]

1. **(Binary) Logistic Regression**, abbreviated as **LR** is a single node perceptron-like classifier, but with....
   - *sign* $\left((\mathbf{w}^*)^T\phi(x)\right)$ replaced by $g\left((\mathbf{w}^*)^T\phi(\mathbf{x})\right)$ where $g(s)$ is sigmoid function: $g(s) = \frac{1}{1+e^{-s}}$
2. $g\left((\mathbf{w}^*)^T\phi(\mathbf{x})\right) = \frac{1}{1+e^{-(\mathbf{w}^*)^T\phi(\mathbf{x})}} \in [0,1]$ can be intepreted as $Pr(y=1|\mathbf{x})$
   - Then $Pr(y=0|x) = ?$ $1 - Pr(y=1|x) = e^{-s}/(1+e^{-s}) = e^{-w^T\phi(x)}/(1+e^{-w^T\phi(x)})$





Logistic Regression: The Sigmoidal (perceptron) Classifier

1. Estimator $\widehat{\mathbf{w}}$ is a function of the dataset $\mathcal{D} = \left\{(\phi(\mathbf{x}^{(1)}, y^{(1)}), (\phi(\mathbf{x}^{(2)}, y^{(2)}), \ldots, (\phi(\mathbf{x}^{(m)}, y^{(m)}))\right\}$
   - Estimator $\widehat{\mathbf{w}}$ is meant to approximate the parameter $\mathbf{w}$.
2. Maximum Likelihood Estimator: Estimator $\widehat{\mathbf{w}}$ that maximizes the likelihood $L(\mathcal{D}; \mathbf{w})$ of the data $\mathcal{D}$.
   - Assumes that all the instances $(\phi(\mathbf{x}^{(1)}, y^{(1)}), (\phi(\mathbf{x}^{(2)}, y^{(2)}), \ldots, (\phi(\mathbf{x}^{(m)}, y^{(m)}))$ in $\mathcal{D}$ are all independent and identically distributed (iid)
   - Thus, Likelihood is the probability of $\mathcal{D}$ under iid assumption: $\hat{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmax}}\ L(\mathcal{D}, \mathbf{w}) =$

$$\prod_{i=1}^{m} \Pr(y^{(i)} \mid x^{(i)}, w) = \prod_{i=1}^{m} \underbrace{\Pr(1 \mid x^{(i)}, w)^{y^{(i)}} \Pr(0 \mid x^{(i)}, w)^{(1-y^{(i)})}}_{\text{Exactly as in Bernoulli}}$$

$$\text{Ber}\left(\frac{1}{1+e^{-w^T\phi(x)}}\right)$$

Logistic Regression: The Sigmoidal (perceptron) Classifier

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   - Thus, Likelihood is the probability of $\mathcal{D}$ under iid assumption: $\hat{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmax}}\ L(\mathcal{D}, \mathbf{w}) =$ $\text{argmax}_{\mathbf{w}} \prod_{i=1}^{m} p(y^{(i)} | \phi(\mathbf{x}^{(i)})) = \text{argmax}_{\mathbf{w}} \prod_{i=1}^{m} \left(\underbrace{\frac{1}{1+e^{-(w)^T\phi(x^{(i)})}}}\right)^{y^{(i)}} \left(\underbrace{\frac{e^{-(w)^T\phi(x^{(i)})}}{1+e^{-(w)^T\phi(x^{(i)})}}}\right)^{1-y^{(i)}}$

Training LR

1. Thus, Maximum Likelihood Estimator for $\mathbf{w}$ is

$$\hat{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmax}}\ L(\mathcal{D}, w) = \underset{\mathbf{w}}{\text{argmax}} \prod_{i=1}^{m} p(y^{(i)}|\phi(\mathbf{x}^{(i)}))$$

$$= \underset{\mathbf{w}}{\text{argmax}} \prod_{i=1}^{m} \left(\frac{1}{1+e^{-\mathbf{w}^T\phi(\mathbf{x}^{(i)})}} \right)^{y^{(i)}} \left(\frac{e^{-\mathbf{w}^T\phi(\mathbf{x}^{(i)})}}{1+e^{-\mathbf{w}^T\phi(\mathbf{x}^{(i)})}} \right)^{1-y^{(i)}}$$

$$= \underset{\mathbf{w}}{\text{argmax}} \prod_{i=1}^{m} \left(f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right)\right)^{y^{(i)}} \left(1 - f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right)\right)^{1-y^{(i)}}$$

$\Pr(1 | x^{(i)}, w)$

2. Maximizing the likelihood $\Pr(\mathcal{D}; \mathbf{w})$ w.r.t $\mathbf{w}$, is the same as minimizing the negative log-likelihood $E(\mathbf{w}) = -\frac{1}{m} \log \Pr(\mathcal{D}; \mathbf{w})$ w.r.t $\mathbf{w}$.
   - Derive the expression for $E(\mathbf{w})$. $\longrightarrow$ H/W
   - $E(\mathbf{w})$ is called the cross-entropy loss function

Distance between 2 probability distributions