Lecture 18: Logistic Regression, Neural Networks

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MAP estimation and regularized LR

1. **FROM** MAP for LR: $\tilde{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmax}}\, \Pr(\mathbf{w}) L(\mathcal{D}, \mathbf{w})$

   $= \underset{\mathbf{w}}{\text{argmax}} \frac{1}{\left(\frac{2\pi}{\lambda}\right)^{\frac{m}{2}}} e^{-\frac{\lambda}{2}\|\mathbf{w}\|_2^2} \prod_{i=1}^{m} \left(f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right)\right)^{y^{(i)}} \left(1 - f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right)\right)^{1-y^{(i)}}$

   .....Taking $-\frac{1}{m}\log(.)$ transformation,
2. **TO** Min of the Regularized Logistic (Cross-Entropy) Loss function:

$$\tilde{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmin}} - \left[\frac{1}{m}\sum_{i=1}^{m}\left(y^{(i)} \log f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right)\right)\right)\right] + \frac{\lambda}{2}\|\mathbf{w}\|_2^2 \tag{1}$$

where we have ignored $-\frac{1}{m}\log\left(\left(\frac{2\pi}{\lambda}\right)^{\frac{m}{2}}\right)$ since this term is independent of $\mathbf{w}$.

.....Thus, MAP $\tilde{\mathbf{w}}$ can be found by minimizing the *Regularized Cross Entropy Error*

Gradient descent for Regularized LR

1. The final descent update

$$-\eta \nabla E(\mathbf{w}) = \eta \left[\frac{1}{m} \sum_{i=1}^{m} \left(y^{(i)} - f_{\mathbf{w}} \left(\mathbf{x}^{(i)} \right) \right) \phi(\mathbf{x}^{(i)}) - \lambda \mathbf{w} \right] \tag{2}$$

2. The iterative update rule:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{k} + \eta \left[\frac{1}{m} \sum_{i=1}^{m} \left(y^{(i)} - f_{\mathbf{w}^k} \left(\mathbf{x}^{(i)} \right) \right) \phi(\mathbf{x}^{(i)}) - \lambda \mathbf{w}^{k} \right] \tag{3}$$

3. Stochastic version of the same:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{k} + \eta \left(y^{(i)} - f_{\mathbf{w}^k} \left(\mathbf{x}^{(i)} \right) \right) \phi(\mathbf{x}^{(i)}) - \eta \lambda \mathbf{w}^{k} \tag{4}$$

One Extension to multi-class logistic

1. Each class $c = 1, 2, \ldots, K-1$ can have a different weight vector $[\mathbf{w}_{c,1}, \mathbf{w}_{c,2}, \ldots, \mathbf{w}_{c,k}, \ldots, \mathbf{w}_{c,p}]$ and

$$p(Y = c|\phi(\mathbf{x})) = \frac{e^{-(\mathbf{w}_c)^T\phi(\mathbf{x})}}{1 + \sum_{k=1}^{K-1} e^{-(\mathbf{w}_k)^T\phi(\mathbf{x})}}$$

for $c = 1, \ldots, K-1$ so that

$$p(Y = K|\phi(\mathbf{x})) = \frac{1}{1 + \sum_{k=1}^{K-1} e^{-(\mathbf{w}_k)^T\phi(\mathbf{x})}}$$

Alternative (equivalent) extension to multi-class logistic

1. Each class $c = 1, 2, \ldots, K$ can have a different weight vector $[\mathbf{w}_{c,1}, \mathbf{w}_{c,2} \ldots \mathbf{w}_{c,p}]$ and

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for $c = 1, \ldots, K$.

Logistic Regression Kernelized

1. We have already seen (a) Cross Entropy loss and (b) Bayesian interpretation for regularization
2. The Regularized (Logistic) Cross-Entropy Loss function (minimized wrt $\mathbf{w} \in \Re^p$):

$$E(w) = -\left[\frac{1}{m}\sum_{i=1}^{m}\left(y^{(i)}\log f_w\left(\mathbf{x}^{(i)}\right) + \left(1-y^{(i)}\right)\log\left(1-f_{\mathbf{w}}\left(\mathbf{x}^{(i)}\right)\right)\right)\right] + \frac{\lambda}{2m}\|\mathbf{w}\|_2^2 \tag{5}$$

3. Equivalent dual kernelized objective[1]
(minimized wrt $\alpha \in \Re^m$):

[1] http://perso.telecom-paristech.fr/~clemenco/Projets_ENPC_files/kernel-log-regression-svm-boosting.pdf

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3. Equivalent dual kernelized objective[1]
(minimized wrt $\alpha \in \Re^m$):

$$E_D(\alpha) = \left[\sum_{i=1}^{m}\left(\sum_{j=1}^{m} -y^{(i)}K\left(\mathbf{x}^{(i)},\mathbf{x}^{(j)}\right)\alpha_j + \frac{\lambda}{2}\alpha_i K\left(\mathbf{x}^{(i)},\mathbf{x}^{(j)}\right)\alpha_j\right) + \log\left(1+\sum_{j=1}^{m}\alpha_j K\left(\mathbf{x}^{(i)},\mathbf{x}^{(j)}\right)\right)\right] \tag{6}$$

Decision function $f_{\mathbf{w}}(\mathbf{x}) = \dfrac{1}{1+\sum_{j=1}^{m}\alpha_j K\left(\mathbf{x},\mathbf{x}^{(j)}\right)}$

[1] http://perso.telecom-paristech.fr/~clemenco/Projets_ENPC_files/kernel-log-regression-svm-boosting.pdf

Some Tutorial 7 and 8 Questions

- Prove that the Kernelized Logistic Regression form is equivalent to the original Logistic Regression minimum regularized cross entropy form: 2 Hints
- Contrast the Kernelized Logistic Regression with the dual of Support Vector Classifier

Logistic Regression Generalized to CRF

1. Multi-class LR: $c \in [1 \dots K]$ has weight vector $[w_{c,1} \dots w_{c,p}]$

$$\Pr(y = c \mid x) = \frac{e^{-w_c^T \phi(\mathbf{x})}}{\sum_{k=1}^{K} e^{-w_k^T \phi(\mathbf{x})}} =$$

[2] http://www.tzi.de/~edelkamp/lectures/ml/scripts/loglinearcrfs.pdf

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$$\Pr(y = c \mid x) = \frac{e^{-w_c^T \phi(\mathbf{x})}}{\sum_{k=1}^{K} e^{-w_k^T \phi(\mathbf{x})}} = \frac{e^{-\tilde{w}^T \phi(\mathbf{x}, y=c)}}{Z(\mathbf{x}, \tilde{w})}$$

   where $\tilde{w} = [w_{1,1} \dots w_{1,p}, \dots w_{c,1} \dots w_{c,p}, \dots w_{K,1} \dots w_{K,p}]$ and $\phi(\mathbf{x}, y) = [\delta(y,1)\phi(\mathbf{x}), \dots, \delta(y,c)\phi(\mathbf{x}) \dots \delta(y,K)\phi(\mathbf{x})]$ and $\delta(a,b) = 1$ if $a = b$ and $= 0$ otherwise

2. Extended to non-iid inference in Conditional Random Fields[2] with $\mathbf{x} = [\mathbf{x}^{(1)} \dots \mathbf{x}^{(n)}]$ and $\mathbf{y} = [y^{(1)} \dots y^{(n)}]$:

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2. Extended to non-iid inference in Conditional Random Fields[2] with $\mathbf{x} = [\mathbf{x}^{(1)} \dots \mathbf{x}^{(n)}]$ and $\mathbf{y} = [y^{(1)} \dots y^{(n)}]$:

$$\Pr(\mathbf{y} \mid \mathbf{x}) = \frac{e^{-\mathbf{w}^T \phi(\mathbf{x}, \mathbf{y})}}{Z(\mathbf{x}, \mathbf{w})}$$

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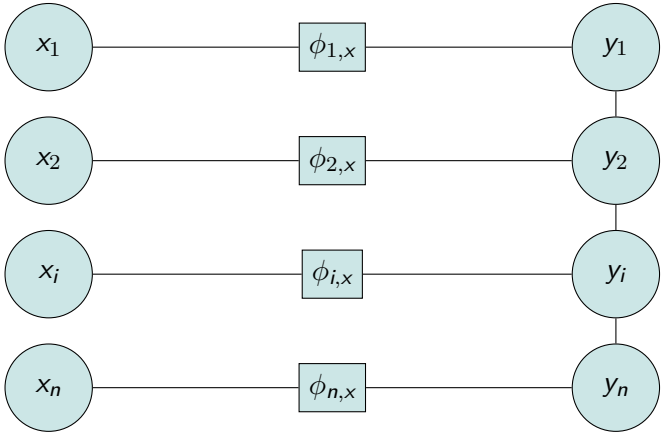
Conditional Random Fields (Linear)[3]

inputs $x-$potentials classes & $y-$potentials $\phi_{i,y}$



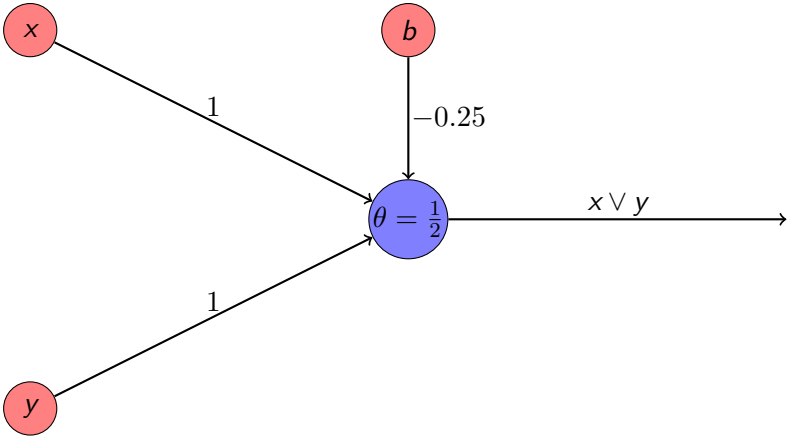
[3] CRF is an instance of Graphical Models (detailed notes at `https://www.cse.iitb.ac.in/~cs725/notes/classNotes/misc/CaseStudyWithProbabilisticModels.pdf`)

Non-linear perceptron?

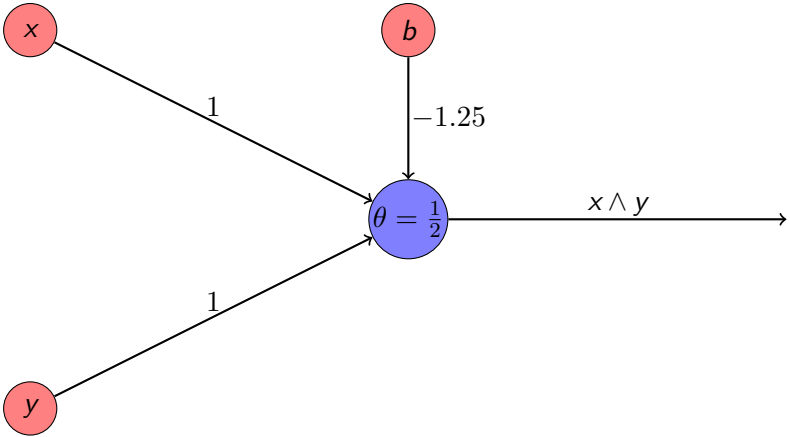
- Kernelized perceptron: $f(x) = sign\left(\sum_i \alpha_i y_i K(x, x_i) + b\right)$
  - INITIALIZE: α=zeroes()
  - REPEAT: for $< x_i, y_i >$
    - ★ If $sign\left(\sum_j \alpha_j y_j K(x_j, x_j) + b\right) \neq y_i$
    - ★ then, $\alpha_i = \alpha_i + 1$
    - ★ endif
- Neural Networks: Cascade of layers of perceptrons giving you non-linearity
  - $sign\left((w^*)^T \phi(x)\right)$ replaced by $g\left((w^*)^T \phi(x)\right)$ where $g(s)$ is a
    1. step function: $g(s) = 1$ if $s \in [\theta, \infty)$ and $g(s) = 0$ otherwise OR
    2. sigmoid function: $g(s) = \frac{1}{1+e^{-s}}$
    3. Rectified Linear Unit (ReLU): $g(s) = max(0, s)$: Most popular activation function
    4. Softplus: $g(s) = \ln(1 + e^s)$

    Options 2, 3 and 4 have the thresholding step deferred. Threshold changes as bias is changed.

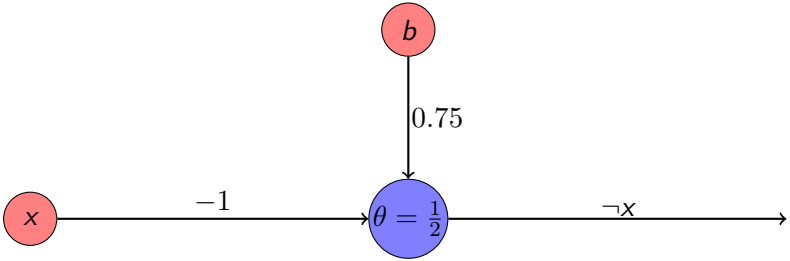
OR using (step) perceptron



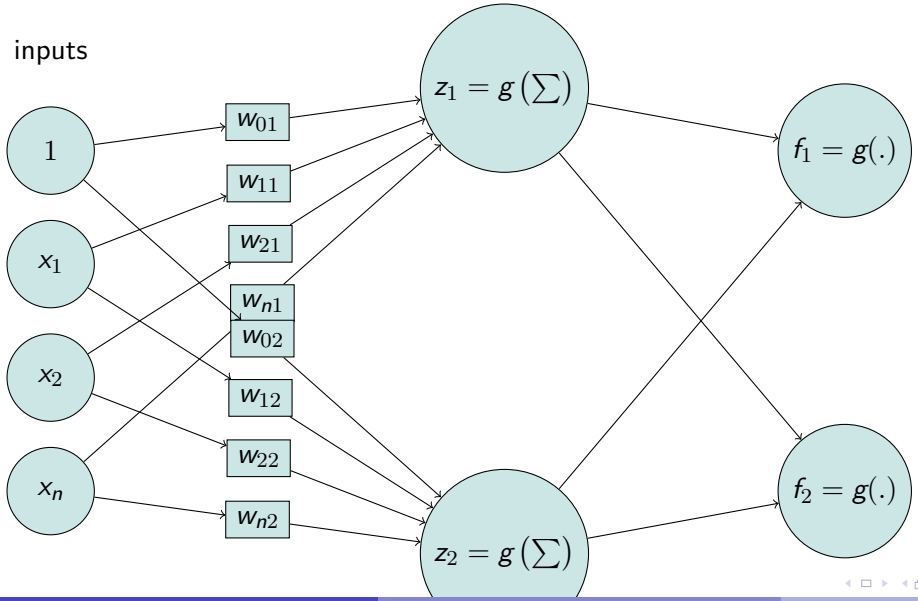
AND using (step) perceptron



NOT using perceptron



Feed-forward Neural Nets



Eg: Feed-forward Neural Net for XOR ($\theta = 0$)