14.6. Principal Curvatures, Gaussian Curvature, Mean Curvature

We will now study how the normal curvature at a point varies when a unit tangent vector varies.

In general, we will see that the normal curvature has a maximum value κ_1 and a minimum value κ_2, and that the corresponding directions are orthogonal. This was shown by Euler in 1760.

The quantity $K = \kappa_1\kappa_2$ called the Gaussian curvature and the quantity $H = (\kappa_1 + \kappa_2)/2$ called the mean curvature, play a very important role in the theory of surfaces.

We will compute H and K in terms of the first and the second fundamental form. We also classify points on a surface according to the value and sign of the Gaussian curvature.

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Recall that given a surface X and some point p on X, the vectors X_u, X_v form a basis of the tangent space $T_p(X)$.

Given a unit vector $\overrightarrow{t} = X_u x + X_v y$, the normal curvature is given by

$$\kappa_N(\overrightarrow{t}) = Lx^2 + 2Mxy + Ny^2,$$

since $Ex^2 + 2Fxy + Gy^2 = 1$.

Usually, (X_u, X_v) is not an orthonormal frame, and it is useful to replace the frame (X_u, X_v) with an orthonormal frame.

One verifies easily that the frame $(\overrightarrow{e_1}, \overrightarrow{e_2})$ defined such that

$$\overrightarrow{e_1} = \frac{X_u}{\sqrt{E}}, \quad \overrightarrow{e_2} = \frac{EX_v - FX_u}{\sqrt{E(EG - F^2)}}.$$

is indeed an orthonormal frame.

With respect to this frame, every unit vector can be written as $\overrightarrow{t} = \cos\theta\,\overrightarrow{e_1} + \sin\theta\,\overrightarrow{e_2}$, and expressing $(\overrightarrow{e_1}, \overrightarrow{e_2})$ in terms of X_u and X_v, we have

$$\overrightarrow{t} = \left(\frac{w\cos\theta - F\sin\theta}{w\sqrt{E}}\right) X_u + \frac{\sqrt{E}\sin\theta}{w} X_v,$$

where $w = \sqrt{EG - F^2}$.

We can now compute $\kappa_N(\overrightarrow{t})$, and we get

$$\kappa_N(\overrightarrow{t}) = L\left(\frac{w\cos\theta - F\sin\theta}{w\sqrt{E}}\right)^2 + 2M\left(\frac{(w\cos\theta - F\sin\theta)\sin\theta}{w^2}\right) + N\,\frac{E\sin^2\theta}{w^2}.$$

We leave as an exercise to show that the above expression can be written as

$$\kappa_N(\overrightarrow{t}) = H + A\cos 2\theta + B\sin 2\theta,$$

where

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)},$$
$$A = \frac{L(EG - 2F^2) + 2EFM - E^2N}{2E(EG - F^2)},$$
$$B = \frac{EM - FL}{E\sqrt{EG - F^2}}.$$

Letting $C = \sqrt{A^2 + B^2}$, unless $A = B = 0$, the function

$$f(\theta) = H + A\cos 2\theta + B\sin 2\theta$$

has a maximum $\kappa_1 = H + C$ for the angles θ_0 and $\theta_0 + \pi$, and a minimum $\kappa_2 = H - C$ for the angles $\theta_0 + \frac{\pi}{2}$ and $\theta_0 + \frac{3\pi}{2}$, where $\cos 2\theta_0 = \frac{A}{C}$ and $\sin 2\theta_0 = \frac{B}{C}$.

The curvatures κ_1 and κ_2 play a major role in surface theory.

Definition 14.6.1 Given a surface X, for any point p on X, letting A, B, H be defined as above, and $C = \sqrt{A^2 + B^2}$, unless $A = B = 0$, the normal curvature κ_N at p takes a maximum value κ_1 and and a minimum value κ_2 called *principal curvatures at* p, where $\kappa_1 = H + C$ and $\kappa_2 = H - C$. The directions of the corresponding unit vectors are called the *principal directions at* p.

The average $H = \frac{\kappa_1 + \kappa_2}{2}$ of the principal curvatures is called the *mean curvature*, and the product $K = \kappa_1 \kappa_2$ of the principal curvatures is called the *total curvature*, or *Gaussian curvature*.

Observe that the principal directions θ_0 and $\theta_0 + \frac{\pi}{2}$ corresponding to κ_1 and κ_2 are orthogonal. Note that

$$K = \kappa_1 \kappa_2 = (H - C)(H + C) = H^2 - C^2 = H^2 - (A^2 + B^2).$$

After some laborious calculations, we get the following (famous) formulae for the mean curvature and the Gaussian curvature:

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)},$$
$$K = \frac{LN - M^2}{EG - F^2}.$$

We showed that the normal curvature κ_N can be expressed as

$$\kappa_N(\theta) = H + A\cos 2\theta + B\sin 2\theta$$

over the orthonormal frame $(\overrightarrow{e_1}, \overrightarrow{e_2})$.

We also showed that the angle θ_0 such that $\cos 2\theta_0 = \frac{A}{C}$ and $\sin 2\theta_0 = \frac{B}{C}$, plays a special role.

Indeed, it determines one of the principal directions.

If we rotate the basis $(\overrightarrow{e_1}, \overrightarrow{e_2})$ and pick a frame $(\overrightarrow{f_1}, \overrightarrow{f_2})$ corresponding to the principal directions, we obtain a particularly nice formula for κ_N. Indeed, since $A = C\cos\ 2\theta_0$ and $B = C\sin 2\theta_0$, letting $\varphi = \theta - \theta_0$, we get

$$\kappa_N(\theta) = \kappa_1\cos^2\varphi + \kappa_2\sin^2\varphi.$$

Thus, for any unit vector $\overrightarrow{t}$ expressed as

$$\overrightarrow{t} = \cos\varphi \overrightarrow{f_1} + \sin\varphi \overrightarrow{f_2}$$

with respect to an orthonormal frame corresponding to the principal directions, the normal curvature $\kappa_N(\varphi)$ is given by *Euler's formula* (1760):

$$\boxed{\kappa_N(\varphi) = \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi.}$$

Recalling that $EG - F^2$ is always strictly positive, we can classify the points on the surface depending on the value of the Gaussian curvature K, and on the values of the principal curvatures κ_1 and κ_2 (or H).

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Definition 14.6.2 Given a surface X, a point p on X belongs to one of the following categories:

(1) *Elliptic* if $LN - M^2 > 0$, or equivalently $K > 0$.

(2) *Hyperbolic* if $LN - M^2 < 0$, or equivalently $K < 0$.

(3) *Parabolic* if $LN - M^2 = 0$ and $L^2 + M^2 + N^2 > 0$, or equivalently $K = \kappa_1 \kappa_2 = 0$ but either $\kappa_1 \neq 0$ or $\kappa_2 \neq 0$.

(4) *Planar* if $L = M = N = 0$, or equivalently $\kappa_1 = \kappa_2 = 0$.

Furthermore, a point p is an *umbilical point* (or *umbilic*) if $K > 0$ and $\kappa_1 = \kappa_2$.

Note that some authors allow a planar point to be an umbilical point, but we don't.

At an elliptic point, both principal curvatures are nonnull and have the same sign. For example, most points on an ellipsoid are elliptic.

At a hyperbolic point, the principal curvatures have opposite signs. For example, all points on the catenoid are hyperbolic.

At a parabolic point, one of the two principal curvatures is zero, but not both. This is equivalent to $K = 0$ and $H \neq 0$. Points on a cylinder are parabolic.

At a planar point, $\kappa_1 = \kappa_2 = 0$. This is equivalent to $K = H = 0$. Points on a plane are all planar points! On a monkey saddle, there is a planar point. The principal directions at that point are undefined.

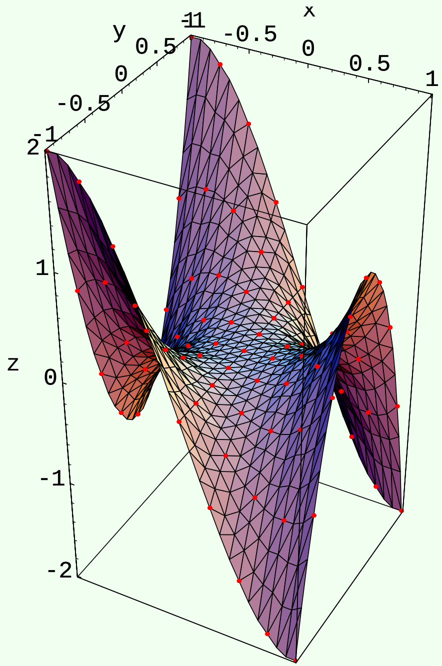

Figure 14.3: A monkey saddle

For an umbilical point, we have $\kappa_1 = \kappa_2 \neq 0$.

This can only happen when $H - C = H + C$, which implies that $C = 0$, and since $C = \sqrt{A^2 + B^2}$, we have $A = B = 0$.

Thus, for an umbilical point, $K = H^2$.

In this case, the function κ_N is constant, and the principal directions are undefined. All points on a sphere are umbilics. A general ellipsoid (a, b, c pairwise distinct) has four umbilics.

It can be shown that a connected surface consisting only of umbilical points is contained in a sphere.

It can also be shown that a connected surface consisting only of planar points is contained in a plane.

A surface can contain at the same time elliptic points, parabolic points, and hyperbolic points. This is the case of a torus.

The parabolic points are on two circles also contained in two tangent planes to the torus (the two horizontal planes touching the top and the bottom of the torus on the following picture).

The elliptic points are on the outside part of the torus (with normal facing outward), delimited by the two parabolic circles.

The hyperbolic points are on the inside part of the torus (with normal facing inward).

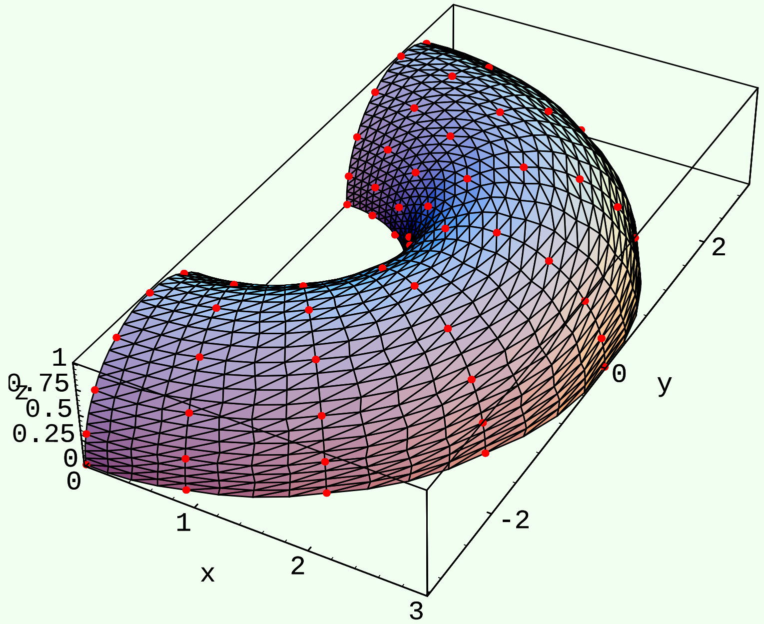

Figure 14.4: Portion of torus

The normal curvature

$$\kappa_N(X_u x + X_v y) = Lx^2 + 2Mxy + Ny^2$$

will vanish for some tangent vector $(x, y) \neq (0, 0)$ iff $M^2 - LN \geq 0$.

Since

$$K = \frac{LN - M^2}{EG - F^2},$$

this can only happen if $K \leq 0$.

If $L = N = 0$, then there are two directions corresponding to X_u and X_v for which the normal curvature is zero.

If $L \neq 0$ or $N \neq 0$, say $L \neq 0$ (the other case being similar), then the equation $L\left(\frac{x}{y}\right)^2 + 2M\frac{x}{y} + N = 0$ has two distinct roots iff $K < 0$.

The directions corresponding to the vectors $X_u x + X_v y$ associated with these roots are called the *asymptotic directions at* p.

These are the directions for which the normal curvature is null at p.

There are surfaces of constant Gaussian curvature. For example, a cylinder or a cone is a surface of Gaussian curvature $K = 0$.

A sphere of radius R has positive constant Gaussian curvature $K = \frac{1}{R^2}$.

Perhaps surprisingly, there are other surfaces of constant positive curvature besides the sphere.

There are surfaces of constant negative curvature, say $K = -1$. A famous one is the *pseudosphere*, also known as *Beltrami's pseudosphere*.

This is the surface of revolution obtained by rotating a curve known as a *tractrix* around its asymptote. One possible parameterization is given by:

$$x = \frac{2\cos v}{e^u + e^{-u}},$$
$$y = \frac{2\sin v}{e^u + e^{-u}},$$
$$z = u - \frac{e^u - e^{-u}}{e^u + e^{-u}},$$

over $]0, 2\pi[\, \times \mathbb{R}$.

The pseudosphere has a circle of singular points (for $u = 0$). The figure below shows a portion of pseudosphere.

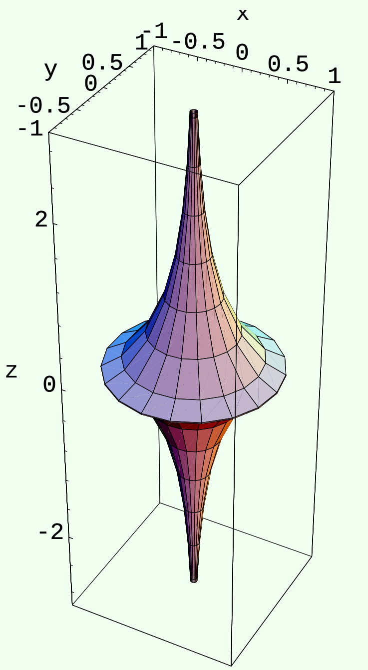

Figure 14.5: A pseudosphere

Again, perhaps surprisingly, there are other surfaces of constant negative curvature.

The Gaussian curvature at a point (x, y, x) of an ellipsoid of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has the beautiful expression

$$K = \frac{p^4}{a^2b^2c^2},$$

where p is the distance from the origin $(0, 0, 0)$ to the tangent plane at the point (x, y, z).

There are also surfaces for which $H = 0$. Such surfaces are called *minimal surfaces*, and they show up in physics quite a bit.

It can be verified that both the helicoid and the catenoid are minimal surfaces.

The Enneper surface is also a minimal surface.

We will see shortly how the classification of points on a surface can be explained in terms of the Dupin indicatrix.

The idea is to dip the surface in water, and to watch the shorlines formed in the water by the surface in a small region around a chosen point, as we move the surface up and down very gently.

But first, we introduce the Gauss map, i.e. we study the variations of the normal $\mathbf{N}_p$ as the point p varies on the surface.