Parameterization

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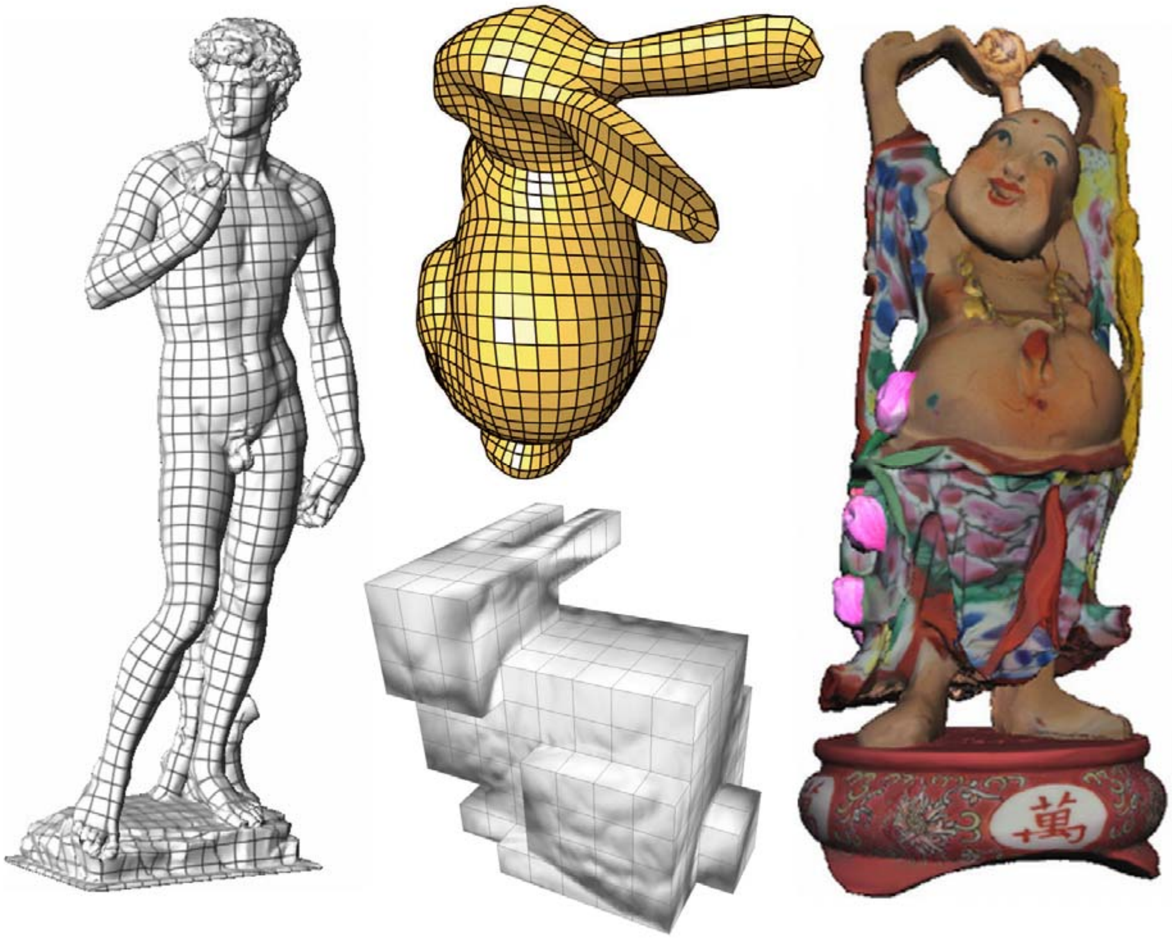
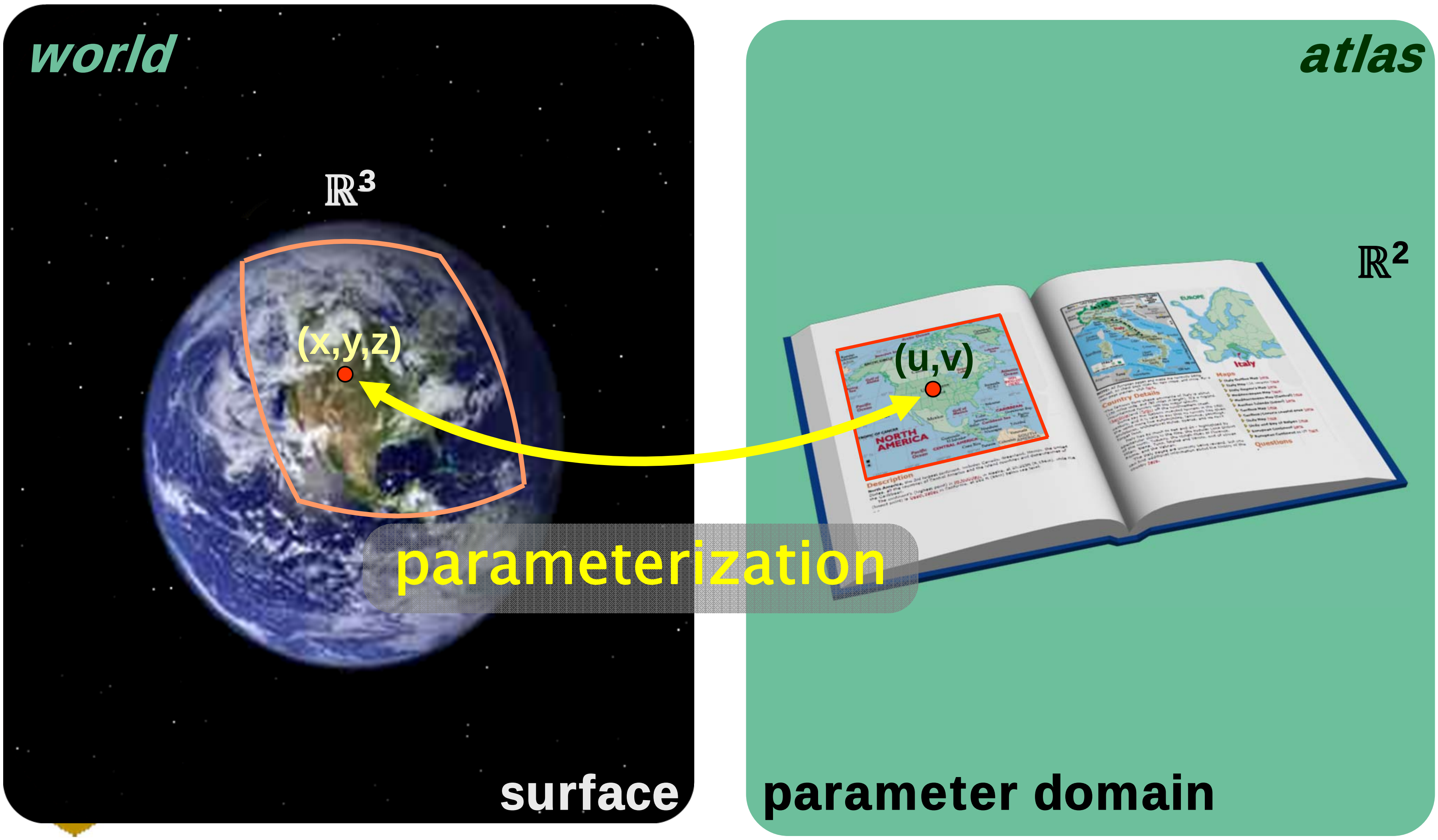
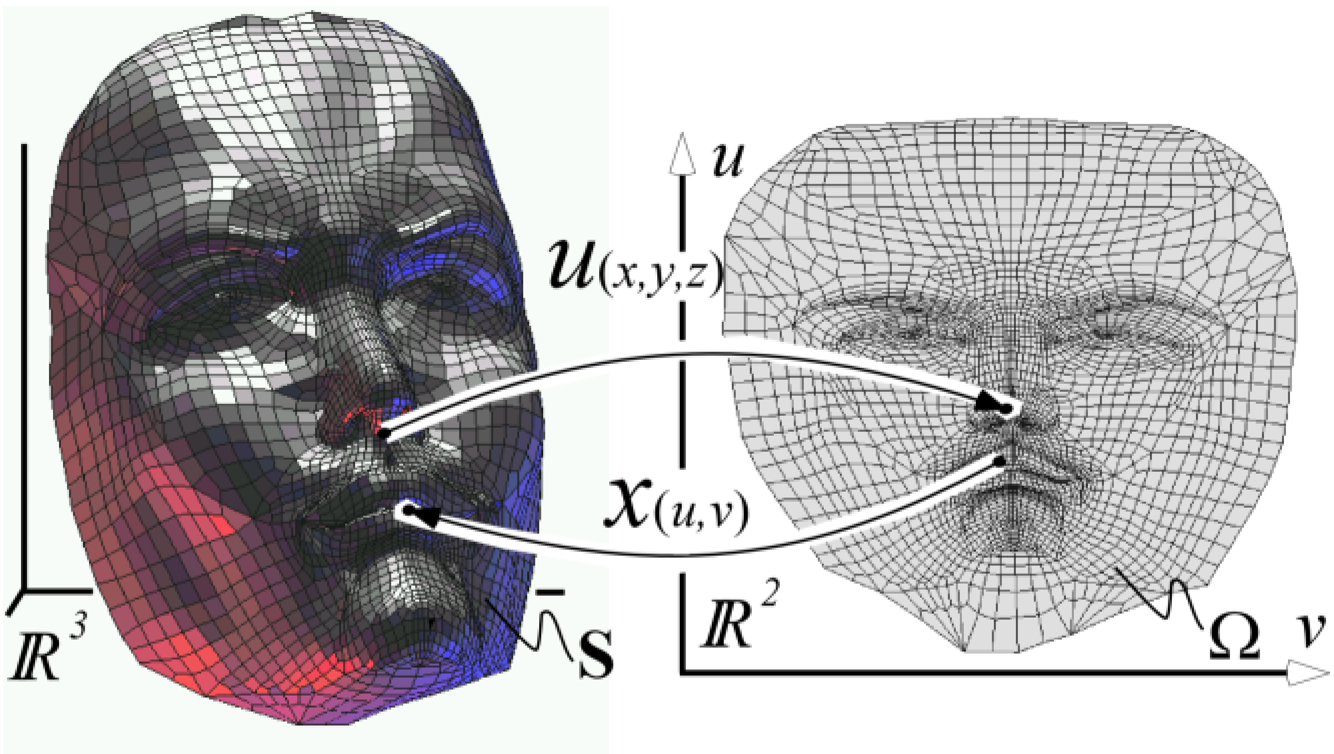


What is Parameterization?



Problem Definition

- Given a surface (mesh) S in R^3 and a domain D find $F: D \leftrightarrow S$ (one-to-one)



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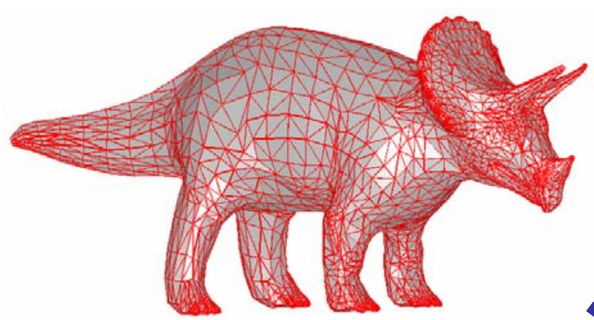
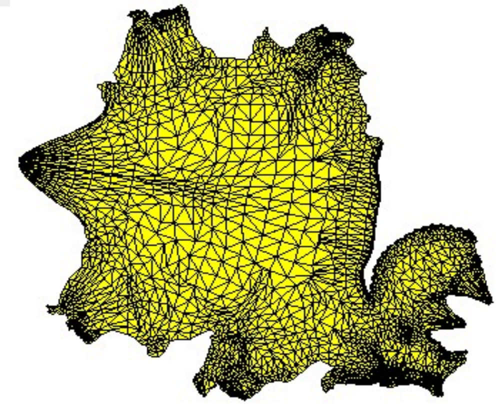
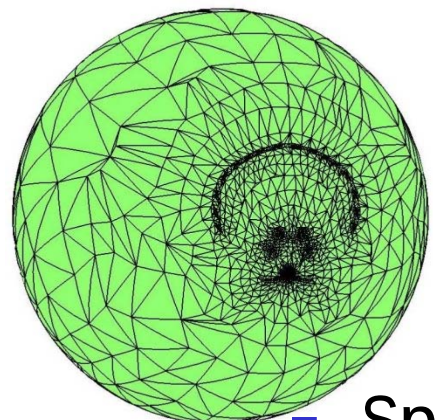
Typical domains:





- Sub-domain of R^2
  - genus zero + **boundary**



- Sphere
  - closed genus zero

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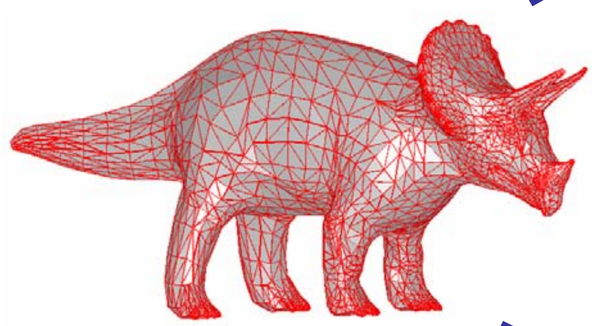
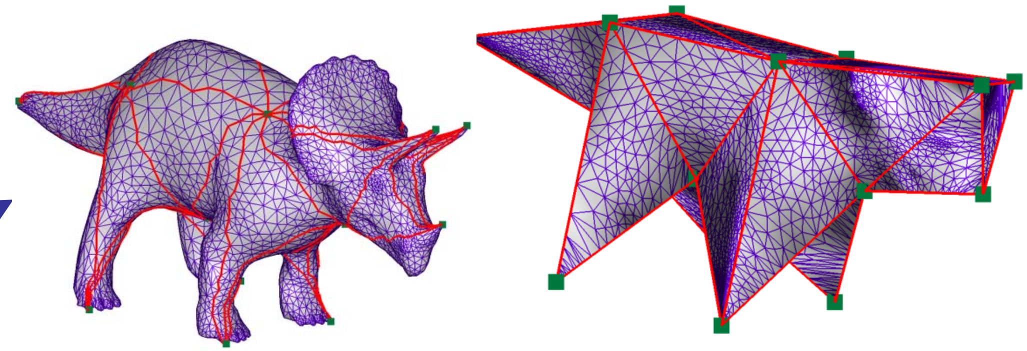
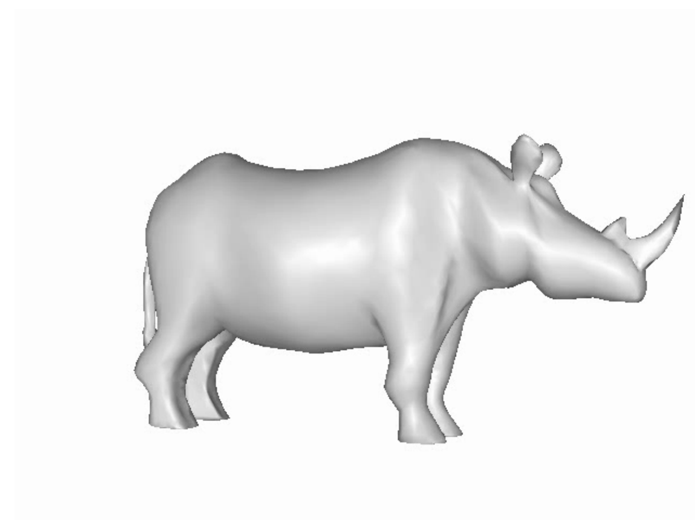
Typical domains:





- Base Mesh
  - all (closed) models



- Cross-Parameterization/ Intersurface Mapping
  - all (closed) models
  - usually utilize common base

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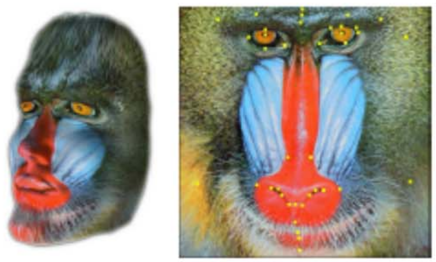
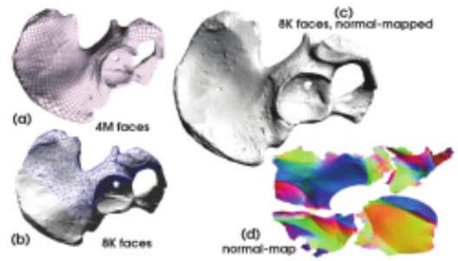
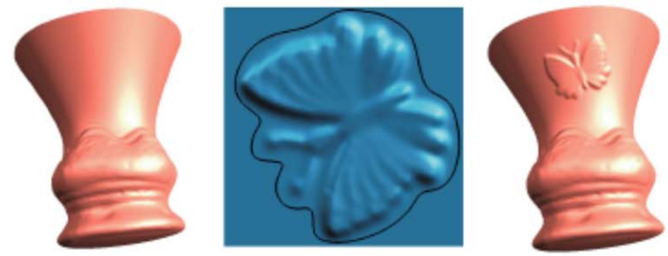
Why Do We Need It?



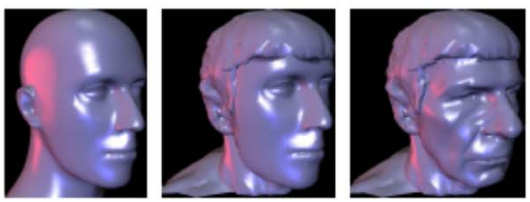
Texture Mapping



Normal Mapping

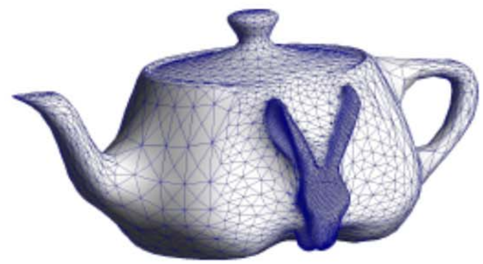


Detail Transfer

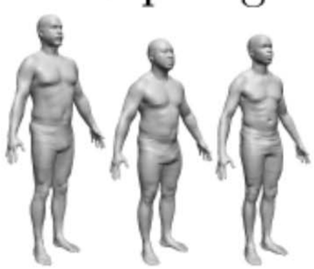


Morphing

Mesh Completion

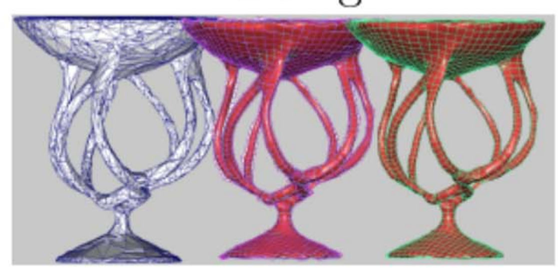


Editing



Databases

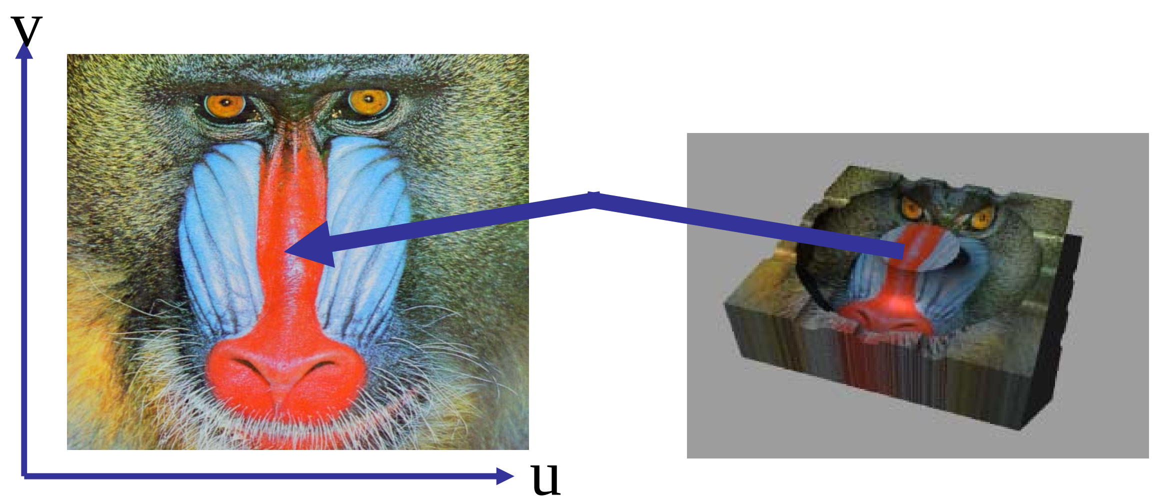
Remeshing



Surface Fitting

Texture Mapping

- Define color for each point on object surface
- Map 2D texture to model surface:
  - Texture pattern defined over 2D domain (u,v)
  - Assign (u,v) coordinates to each point on surface



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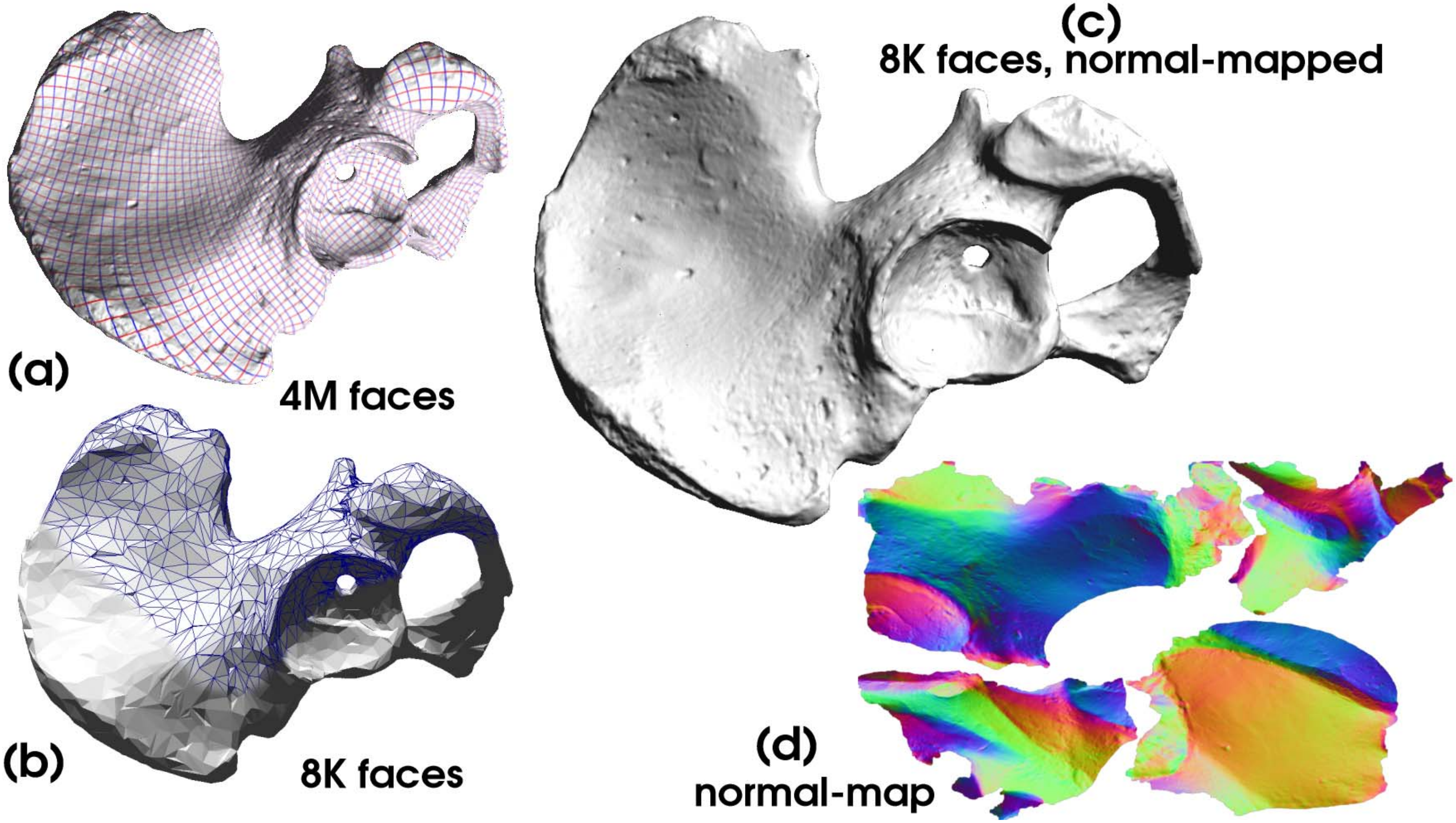
Normal/Bump mapping



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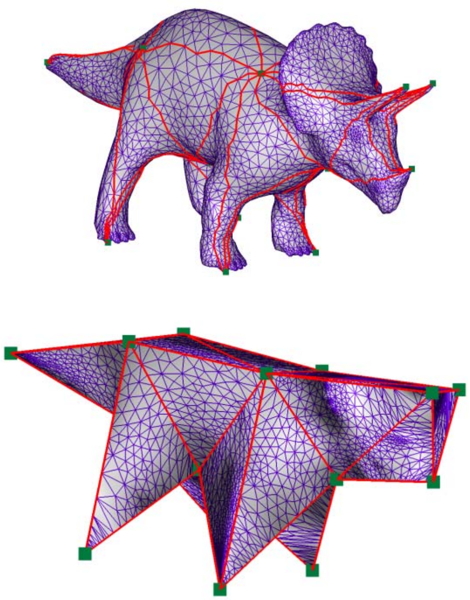
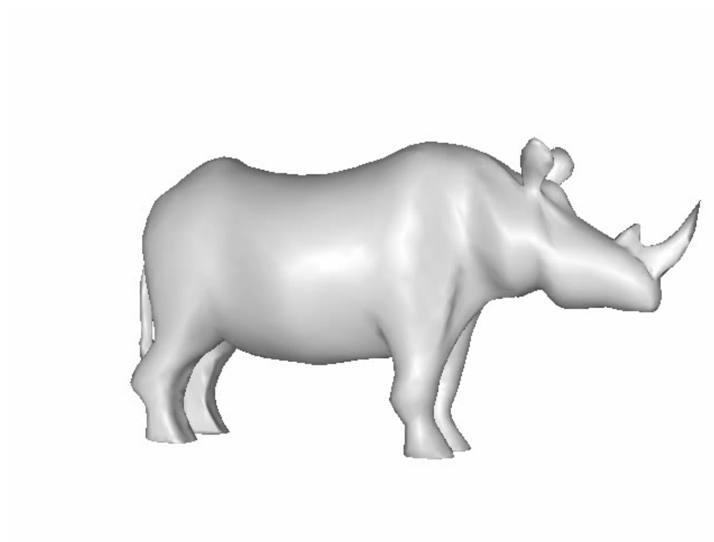
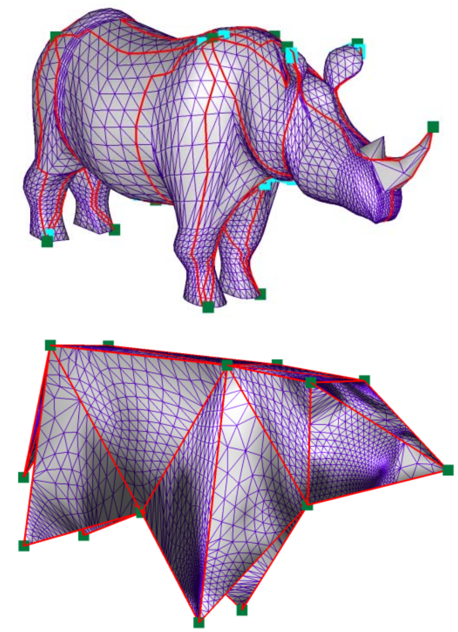
Morphing/Properties Transfer

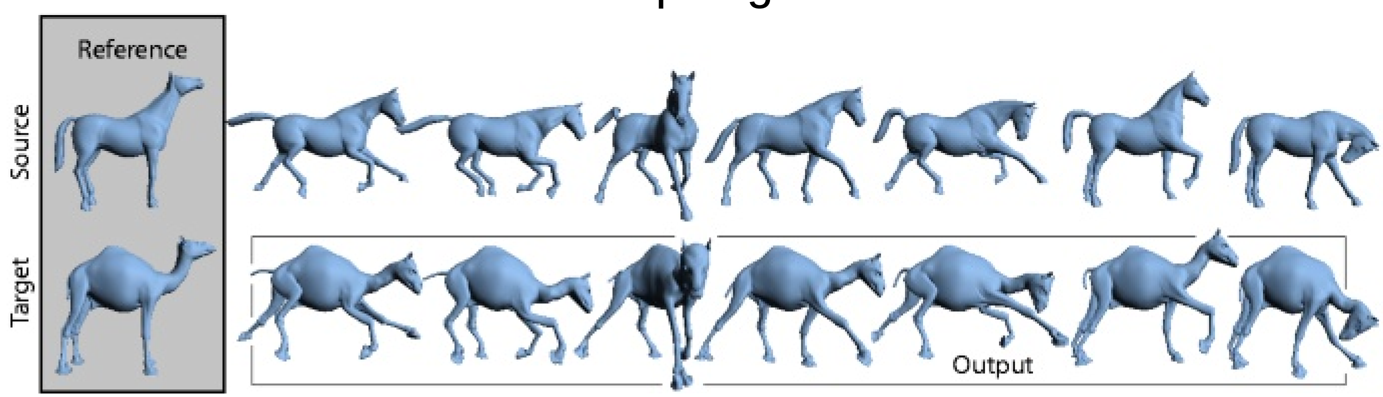
- Require cross-parameterization







Morphing



Deformation Transfer

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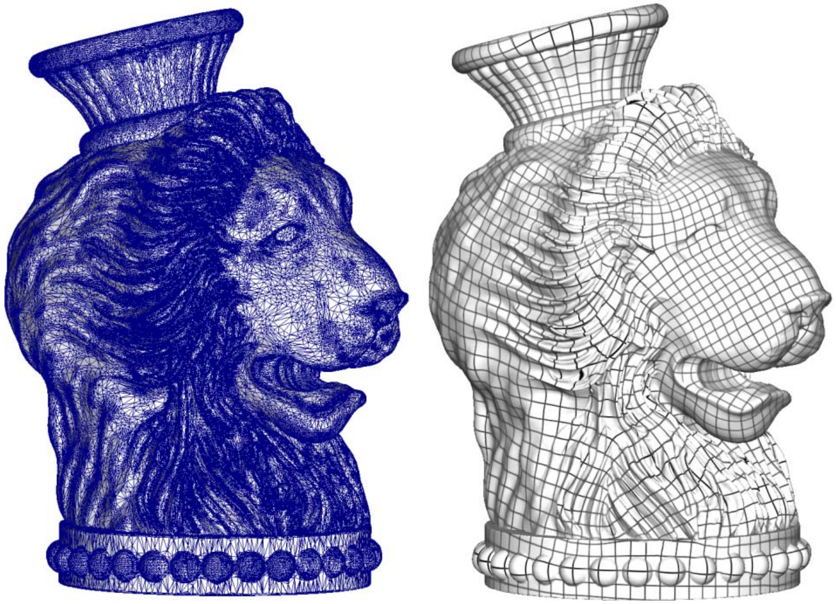
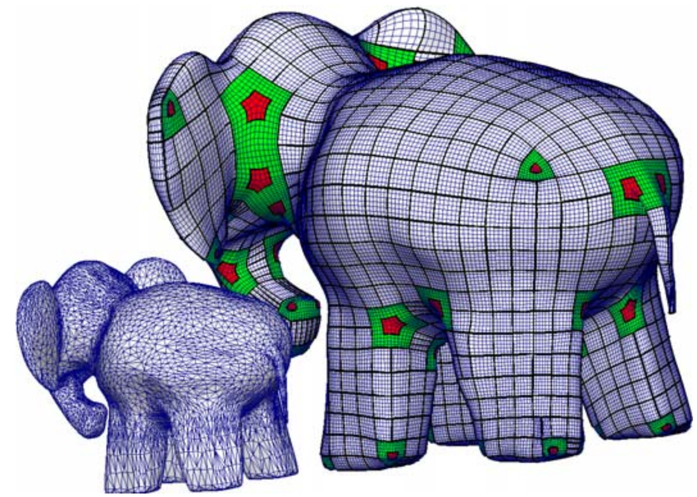
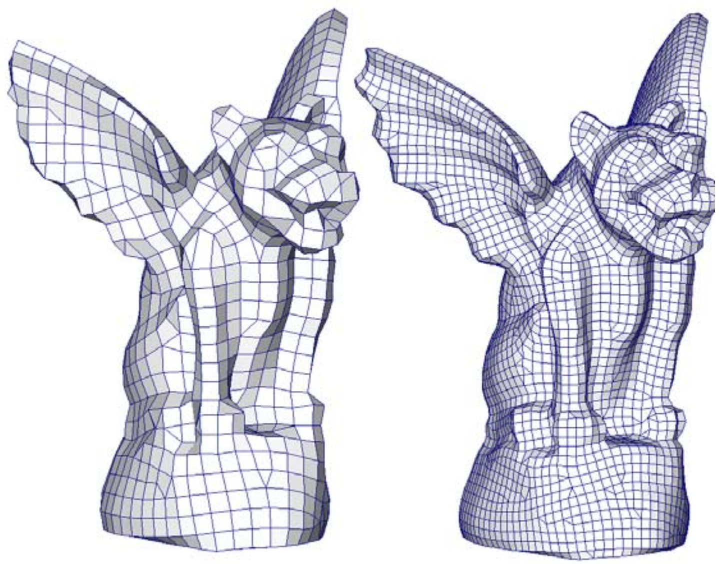
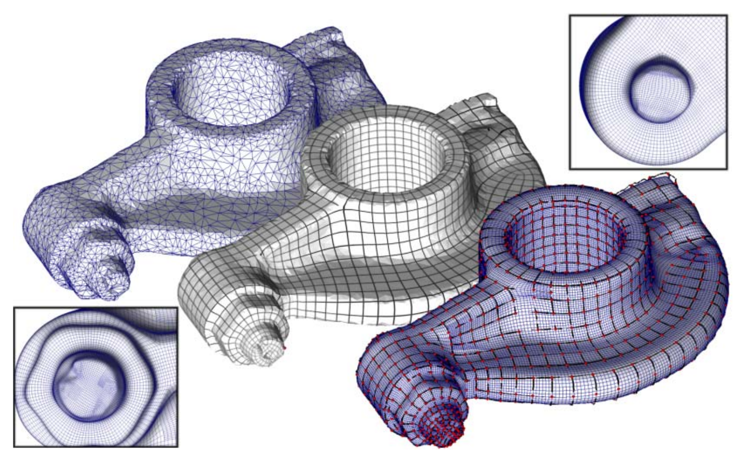
Remeshing & Surface Fitting









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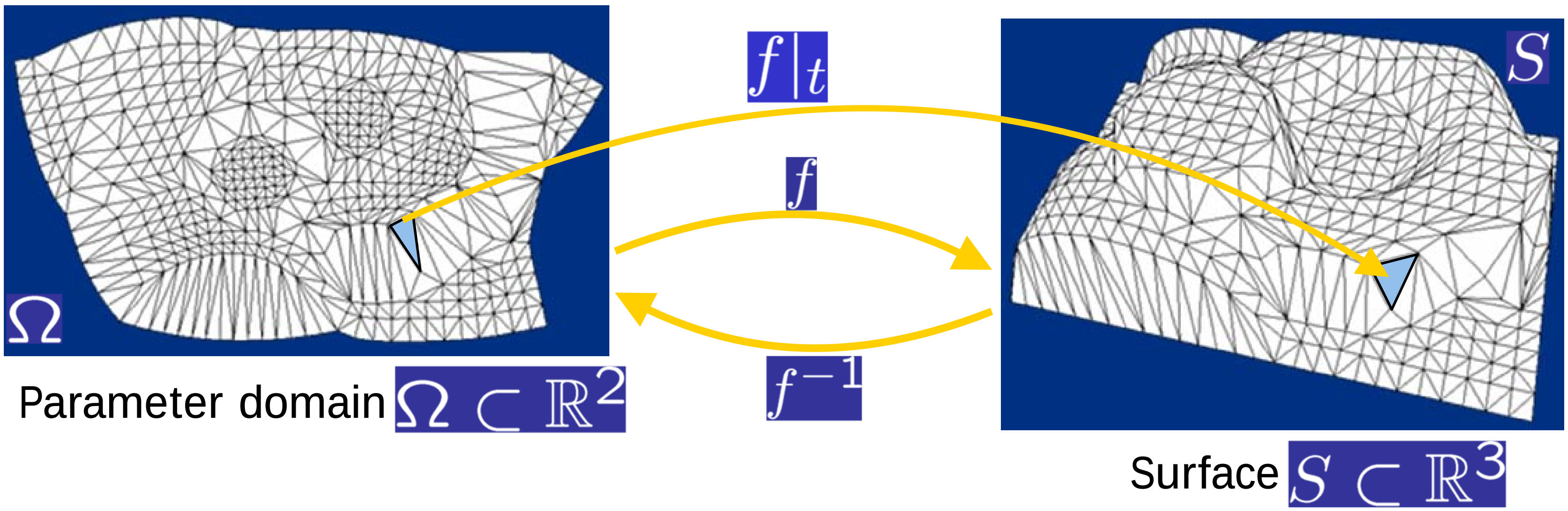
Theory/Background

Parameterization



- Parameterization $f : \Omega \longrightarrow S$
  - f is piecewise linear
    - $f|_t$ is linear (barycentric)
  - f is bijective
    - at least locally

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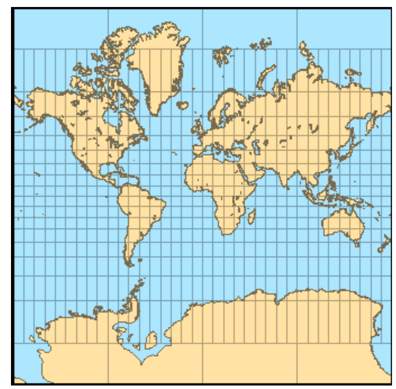
Example – Mappings of the Earth

- Usually, surface properties get **distorted**

orthographic
~ 500 B.C.

stereographic
~ 150 B.C.



Mercator
1569

Lambert
1772

conformal
(angle-preserving)

equiareal
(area-preserving)

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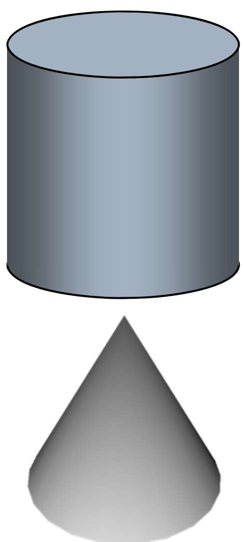
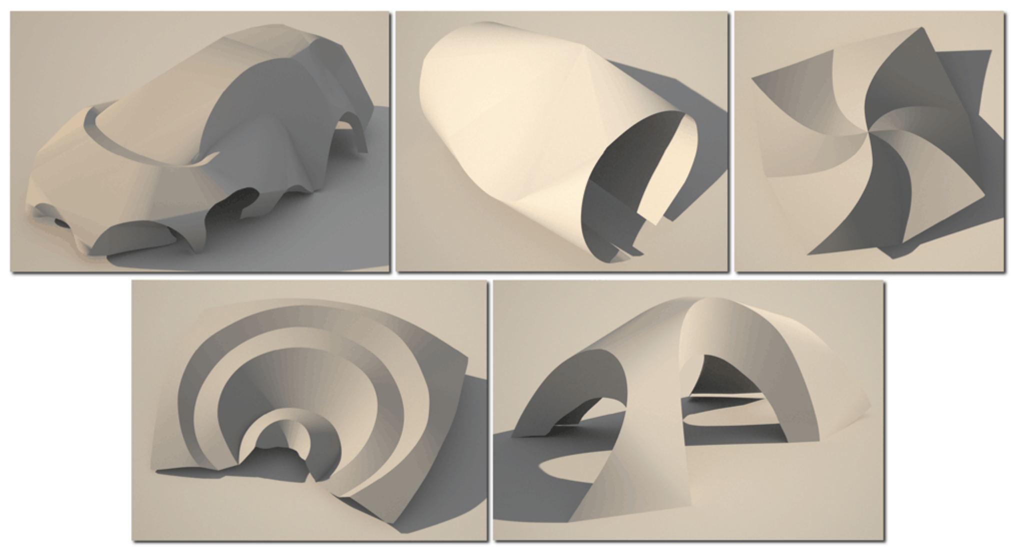
Distortion is (almost) Inevitable

- **Theorema Egregium** (C. F. Gauss)

  "*A general surface cannot be parameterized without distortion.*"
- no distortion = conformal + equiareal = **isometric**
- requires surface to be **developable**





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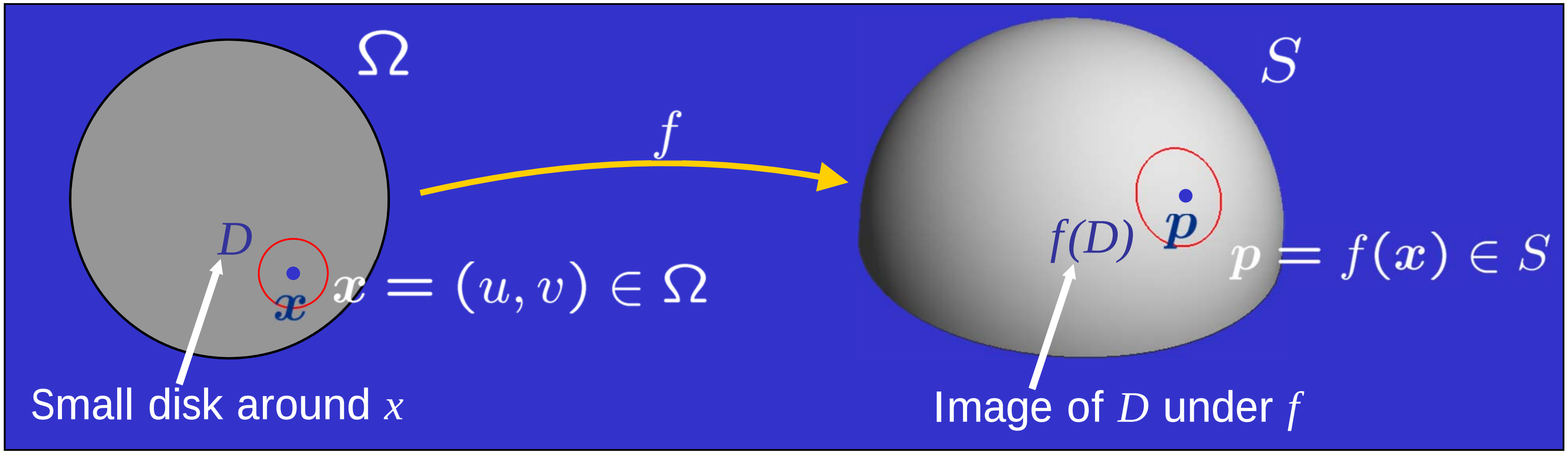
What is Distortion?



- Distortion (at x): How different is $f(D)$ from D
  - How to measure?

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Linearization

- Use Taylor expansion of f

$$f(y) = f(x) + J_f(y - x) + \ldots$$

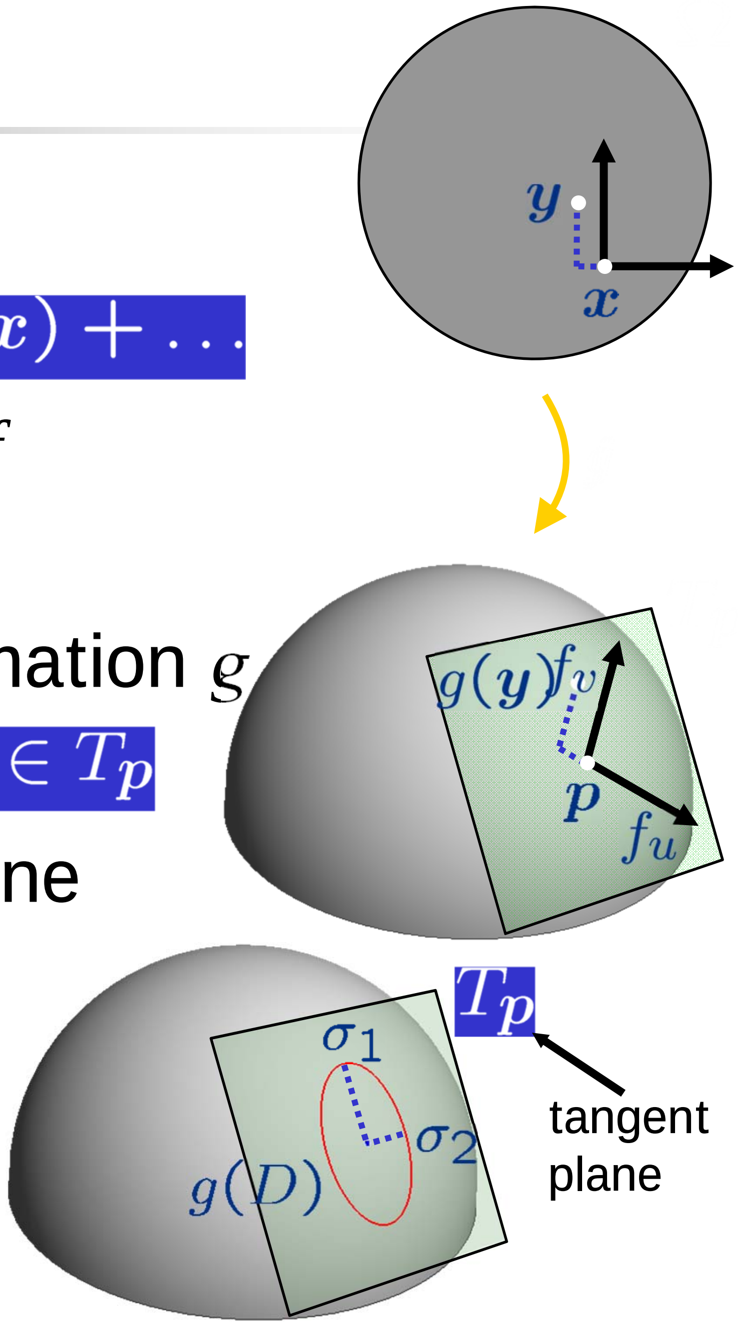
  where J_f is the Jacobian of f

$$J_f = [f_u, f_v] \in \mathbb{R}^{3\times 2}$$

- Replace f by linear approximation g

$$g(y) = p + J_f(y - x) \in T_p$$

- $g(D)$ – *ellipse* in tangent plane
  - semiaxes $r\,\sigma_1$ $\quad$ $r\,\sigma_2$



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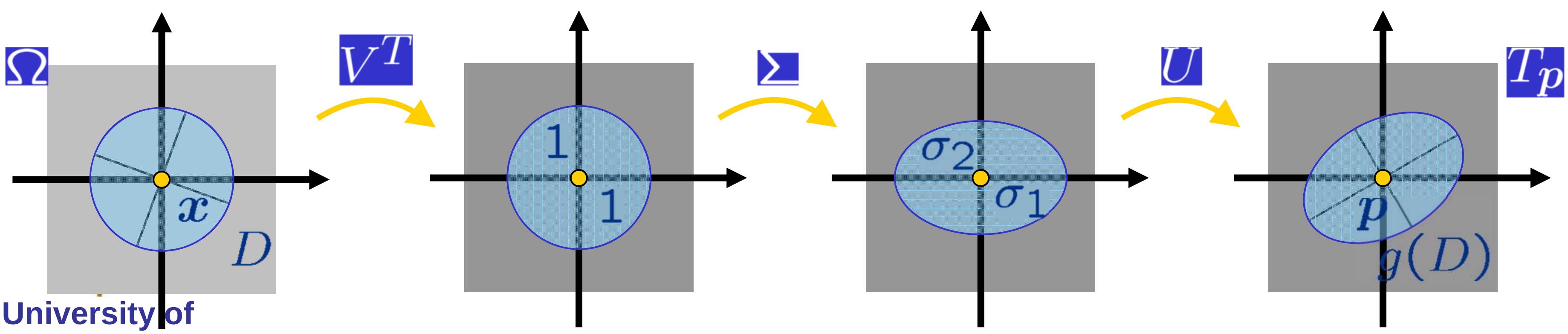
Linear Map Surgery

- **Singular Value Decomposition** (SVD) of J_f

$$J_f = U\,\Sigma\,V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

with *rotations* $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$
and *scale factors* (singular values)

$$\sigma_1 \geq \sigma_2 > 0$$



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Notion of Distortion

- **isometric** or **length**-preserving

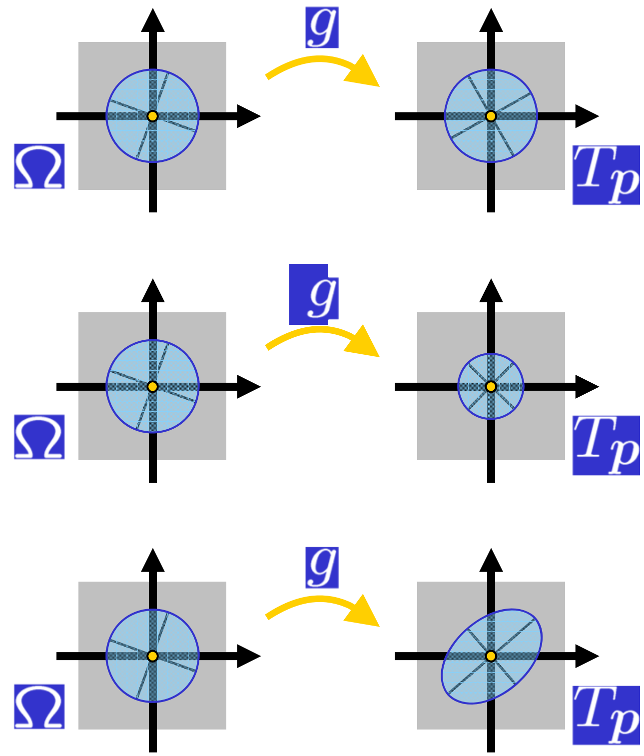
  $\sigma_1 = \sigma_2 = 1$

- **conformal** or **angle**-preserving

  $\sigma_1 = \sigma_2$

- **equiareal** or **area**-preserving

  $\sigma_1 \cdot \sigma_2 = 1$



- Everything defined **pointwise** on Ω

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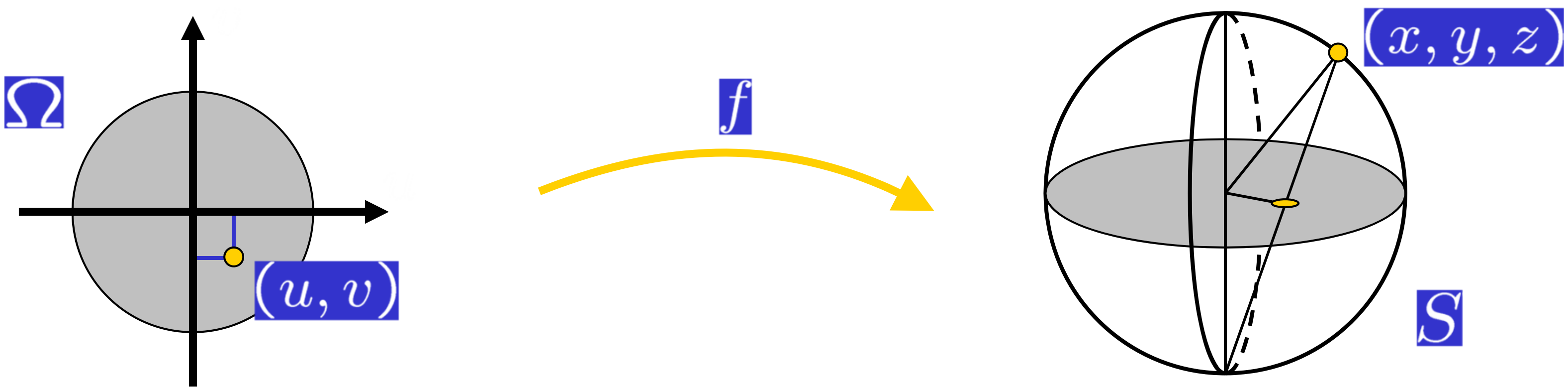
Example – Stereographic Projection



$$f(u, v) = (2ud, 2vd, (1 - u^2 - v^2)d)$$

$$J_f = \begin{pmatrix} 2d - 4u^2d^2 & -4uvd^2 \\ -4uvd^2 & 2d - 4v^2d^2 \\ -4ud^2 & -4vd^2 \end{pmatrix}$$

$$d = \frac{1}{1+u^2+v^2}$$

$\sigma_1 = \sigma_2 = 2d \Rightarrow$ **conformal**

Measuring Distortion

- **Local** distortion measure function of σ_1 and σ_2

$$E : (\mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \mathbb{R}, \qquad (\sigma_1, \sigma_2) \mapsto E(\sigma_1, \sigma_2)$$

- **Overall** distortion

$$E(f) = \int_\Omega E(\sigma_1(u,v), \sigma_2(u,v))\, du\, dv \Big/ \text{Area}(\Omega)$$

- On mesh constant per triangle

$$E(f) = \sum_{t \in \Omega} E(t) A(t) \Big/ \sum_{t \in \Omega} A(t)$$

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Examples – Conformal Measures

- **Conformal** energy

$$E_C = (\sigma_1 - \sigma_2)^2/2$$

[Pinkall & Polthier 1993]
[Lévy et al. 2002]
[Desbrun et al. 2002]

- **MIPS** energy

$$E_M = \kappa_F(J_f) = \|J_f\|_F \, \|J_f{}^{-1}\|_F = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$

[Hormann & Greiner 2000]

- *Rieman theorem:* any C^1 continuous surface in R^3 can be mapped conformally to fixed domain in R^2
  - Nearly true for meshes

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Examples – Stretch

- **Stretch** energies

$$E_2 = \frac{1}{\sqrt{2}} \|J_f\|_F = \sqrt{({\sigma_1}^2 + {\sigma_2}^2)/2}$$

$$E_S = \max(\sigma_1, \frac{1}{\sigma_2})$$

[Sander et al. 2001]
[Sorkine et al. 2002]

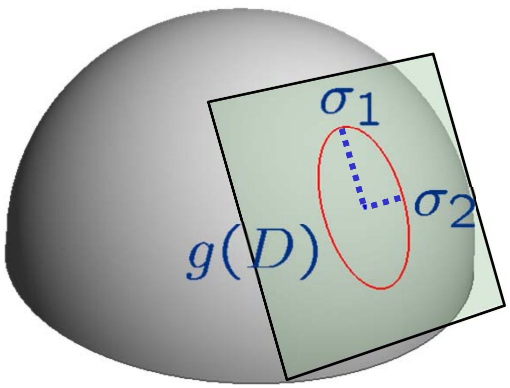
Detailed Example

- Use Taylor expansion to replace f by linear approximation

$$g(y) = p + J_f(y - x) \in T_p$$

where J_f is the Jacobian of f

$$J_f = [f_u, f_v] \in \mathbb{R}^{3\times 2}$$



- Derivation:

Given a triangle T with 2D texture coordinates p_1, p_2, p_3, $p_i = (s_i, t_i)$, and corresponding 3D coordinates q_1, q_2, q_3, the unique affine mapping $S(p) = S(s,t) = q$ is

$$S(p) = \left(\langle p, p_2, p_3\rangle q_1 + \langle p, p_3, p_1\rangle q_2 + \langle p, p_1, p_2\rangle q_3\right) / \langle p_1, p_2, p_3\rangle$$

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Detailed Example

$$S(p) = \left(\langle p, p_2, p_3\rangle q_1 + \langle p, p_3, p_1\rangle q_2 + \langle p, p_1, p_2\rangle q_3\right) / \langle p_1, p_2, p_3\rangle$$

- Jacobian $[S_s, S_t]$:

$$S_s = \partial S/\partial s = \left(q_1(t_2 - t_3) + q_2(t_3 - t_1) + q_3(t_1 - t_2)\right)/(2A)$$

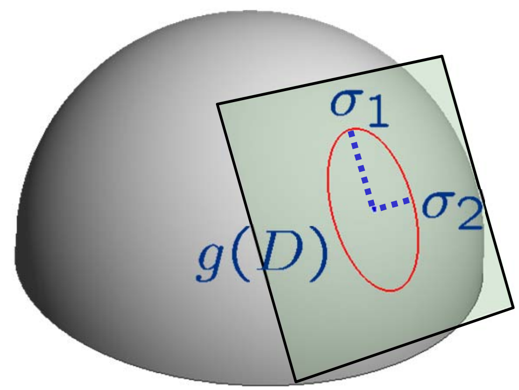
$$S_t = \partial S/\partial t = \left(q_1(s_3 - s_2) + q_2(s_1 - s_3) + q_3(s_2 - s_1)\right)/(2A)$$

$$A = \langle p_1, p_2, p_3\rangle = \left((s_2 - s_1)(t_3 - t_1) - (s_3 - s_1)(t_2 - t_1)\right)/2$$

- Singular values:

$$\sqrt{1/2\left((a+c) + \sqrt{(a-c)^2 + 4b^2}\right)}$$

$$\sqrt{1/2\left((a+c) - \sqrt{(a-c)^2 + 4b^2}\right)}$$



$$a = S_s \cdot S_s, \quad b = S_s \cdot S_t, \quad \text{and} \quad c = S_t \cdot S_t$$

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