Differential Geometry
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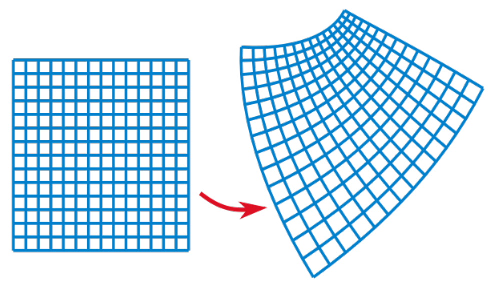
Measuring Lengths – The First Fundamental Form

Patching up the Coordinate Patches. Recall that a proper coordinate patch of a surface is given by parametric equations $\mathbf{x} = (x(u,v), y(u,v), z(u,v))$ such that x, y, z are one-to-one continuous functions with continuous inverses, continuous derivatives and such that $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$. We pointed out that it may not be possible to describe the whole surface with a single coordinate patch but it is always be possible to cover the entire surface by "patching" several different coordinate patches together.

An additional condition that we have to ensure is that two patches must **agree** on the region on which they overlap. This is guaranteed by a condition stating that if $\mathbf{x} = \mathbf{x}(u,v)$ is one coordinate patch defined on domain D and $\bar{\mathbf{x}} = (\bar{u}, \bar{v})$ is another defined on domain $\bar{D}$, that then the composite functions $\mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^{-1} \circ \mathbf{x}$ are one-to-one and onto continuous functions on the intersection of D and $\bar{D}$ with continuous derivatives. If this condition is satisfied, we say that the patches **overlap smoothly**.

This leads us to a more formal definition of a surface. We say that M is a **surface** if there is a collection of coordinate patches such that:



Coordinate patch

1. The coordinate patches cover every point of M and they overlap smoothly.

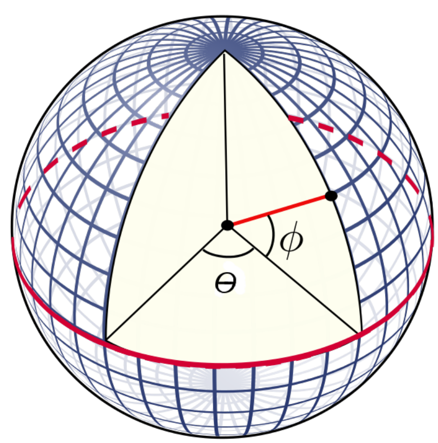
(2) Every two different points on M can be covered by two different patches.

(3) The collection of patches is maximal collection with conditions (1)–(3) (i.e. if a patch overlaps smoothly with every patch in collection is itself in collection).

Let us consider the following examples now.

Examples. (1) Recall that the equation of a sphere of radius a in spherical coordinates is $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$. The angle θ corresponds to the geographical longitude and takes values between π and $-\pi$.

The values of θ from $[-\pi, 0]$ are usually considered as west and are referred to by their absolute value. For example, the longitude of Philadelphia $75°10'$ west corresponds to $\theta = -75°10'$. The values of θ from $[0, \pi]$ are considered to be east. The angle ϕ corresponds to the geographical latitude and takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The values of ϕ from $[-\frac{\pi}{2}, 0]$ are considered as south and are referred to by their absolute value. The values of ϕ from $[0, \frac{\pi}{2}]$ are considered to be north.



Spherical coordinates

For example, the latitude of Philadelphia $39°57'$ north corresponds to $\phi = 39°57'$.

The length of $\frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi}$ is computed to be $a^2|\cos\phi|$. Thus, the cross product is not nonzero if $\cos\phi = 0$. So at point at which $\phi = \frac{\pi}{2}$ which is the north pole and at point at which $\phi = \frac{-\pi}{2}$ which is the south pole. At these two points the geographical longitude is not uniquely defined. Also, if we consider the values of θ to be between $-\pi$ and π, at the meridian at $\theta = \pi$ (the international date line), the mapping is not one-to-one (reflected in the fact that the time on the international date line is both tomorrow and today).

Because of this fact, the coordinate patch given by spherical coordinates is not sufficient to cover the entire sphere. Thus, in addition to the given coordinate patch with $-\pi < \theta < \pi$ and $\frac{-\pi}{2} < \phi < \frac{\pi}{2}$, we may consider another patch given by $0 < \theta < 2\pi$. In this case, the 180°-meridian is covered but the map at Greenwich meridian is not one-to-one. Note that we would still have to add another patches to cover the two poles.

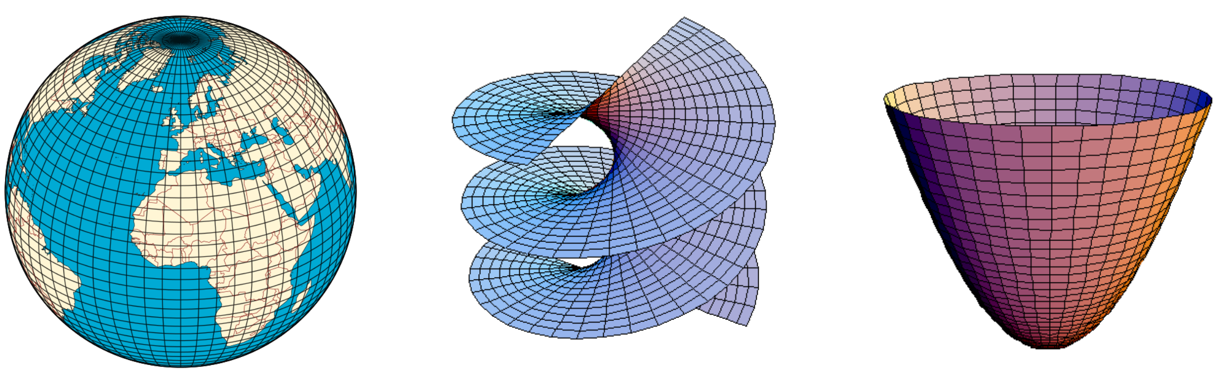
(2) Consider the representation of sphere of radius a as $x^2 + y^2 + z^2 = a^2$. Solving for z, we obtain two coordinate patches $z = \pm\sqrt{a^2 - x^2 - y^2}$. These maps represent proper coordinate patches just over the interior of the circle $x^2 + y^2 = a^2$ so they do not cover equator so we need more patches. Solving for y, we obtain another two patches $y = \pm\sqrt{a^2 - x^2 - z^2}$ that cover the front and the back of the sphere but do not cover the circle $x^2 + z^2 = a^2$. Combining these four patches, we covered everything but the intersections of two circles, the points $(0, a, 0)$ and $(0, -a, 0)$. Finally, to completely cover the sphere, we can patch the two holes with $x = \pm\sqrt{a^2 - y^2 - z^2}$. Thus, we can cover the entire sphere in six proper coordinate patches.

A curve on a surface $\mathbf{x}(u, v)$ is given by $\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t))$. Two important special cases are the following.

- Taking v to be a constant v_0, one obtains the curve $\boldsymbol{\gamma}_1(u) = \mathbf{x}(u, v_0)$. This curve is called a u-**curve.** The velocity vector $\frac{\partial \mathbf{x}}{\partial u}$ is in the tangent plane.
- Taking u to be a constant u_0, one obtains the curve $\boldsymbol{\gamma}_2(v) = \mathbf{x}(u_0, v)$. This curve is called a v-**curve.** The velocity vectors $\frac{\partial \mathbf{x}}{\partial v}$ is in the tangent plane.

  The condition $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$ guarantees that the vectors $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ are a basis of the tangent plane.

Surfaces are often represented by by graphing a mesh determined by u and v curves. For example, this is how graphs in Matlab are obtained.



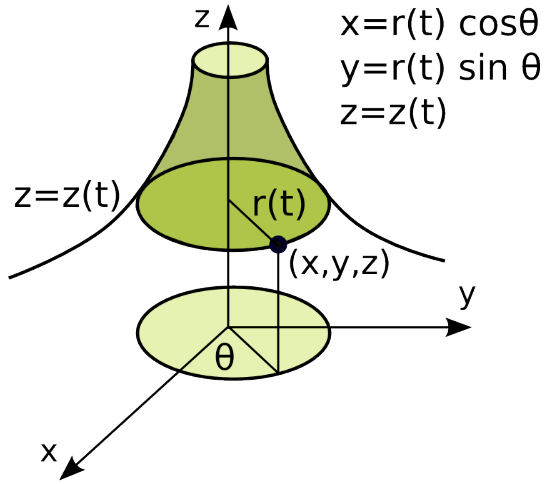
Meridians and Parallels

Examples.

1. On a sphere $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$, the ϕ-curves are circles of constant longitude **meridians** and the θ-curves are circles of constant latitude, **parallels**.

2. **Helicoid**. The surface parametrized by $(r\cos\theta, r\sin\theta, a\theta)$ has θ-curves helices spiraling about cylinder of radius r_0 and r-curves lines $(r\cos\theta_0, r\sin\theta_0, a\theta_0)$ passing through z-axis in a plane parallel to xy-plane. The surface resembles the spiral ramps like those found in garages.

3. **Surfaces of revolution**. A surface of revolution of curve $\alpha = (r(t), z(t))$ in yz-plane (or, equivalently, xz-plane) about z-axis can be given by $(r(t)\cos\theta, r(t)\sin\theta, z(t))$. The θ-curves are circles of radii $r(t_0)$ in horizontal planes passing $z(t_0)$. They are also called **circles of latitude** or **parallels** by analogy with sphere. The t-curves have the same shape as the curve α except that they line in vertical planes at longitude θ_0. They are called **meridians**.



4. The class of surfaces generated by moving a line along some direction are called **ruled surfaces** which we turn to in more details now.

Ruled surfaces can be described by the property that through every point there is a line completely contained in the surface. This line is called a **ruling**. If $\boldsymbol{\alpha}(t)$ is a curve that cuts across all the rulings and $\boldsymbol{\beta}(t)$ is the direction of ruling (you can think that the surface is obtained by moving vector $\boldsymbol{\beta}$ along the curve $\boldsymbol{\alpha}$), the surface can be described by

$$\mathbf{x}(t, s) = \boldsymbol{\alpha}(t) + s\boldsymbol{\beta}(t)$$

The s-curves are ruling lines $\boldsymbol{\alpha}(t_0) + s\boldsymbol{\beta}(t_0)$. In case that $\boldsymbol{\beta}$ is a constant vector, the t-curves represent curve $\boldsymbol{\alpha}$ translated in space.

Hyperboloid as a ruled surface

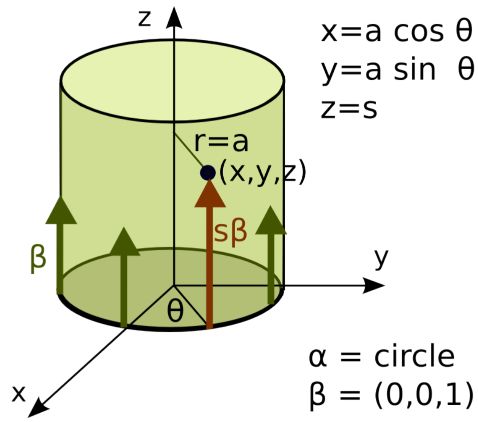
Examples of ruled surfaces.

1. A plane can be considered to be a ruled surface letting $\boldsymbol{\alpha}(t)$ be a line and $\boldsymbol{\beta}(t)$ be a constant vector.

2. A **cone** is a ruled surface with $\boldsymbol{\alpha}$ constant, say point P. The point P is called the **vertex** of the cone. In case that $\boldsymbol{\beta}$ makes a constant angle with fixed line through P (called the **axis**) of the cone), we obtain the **right circular cone**. For example, the cone $z = \sqrt{x^2 + y^2}$ can be parametrized by taking $\boldsymbol{\alpha} = (0, 0, 0)$ and $\boldsymbol{\beta} = (\cos t, \sin t, 1)$ and getting $(s\cos t, s\sin t, s)$.

3. A **cylindrical surface** is defined as a ruled surface with $\boldsymbol{\beta}$ constant vector. If $\boldsymbol{\alpha}$ is a circle, the cylindrical surface is said to be **circular cylinder**. If $\boldsymbol{\beta}$ is a vector perpendicular to the plane of circle $\boldsymbol{\alpha}$ the circular cylinder is said to be **right**.

   For example, the cylinder $x^2 + y^2 = a^2$ can be considered as a ruled surfaces with $\boldsymbol{\alpha}$ being the circle in xy-plane and $\boldsymbol{\beta} = (0, 0, 1)$.

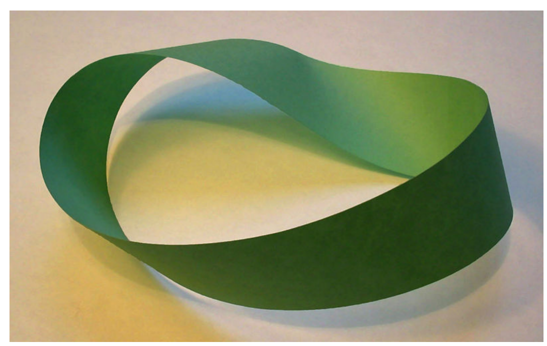


4. The **helicoid** $(r\cos\theta, r\sin\theta, a\theta)$ can be considered to be a ruled surface by taking $\boldsymbol{\alpha} = (0, 0, a\theta)$ and $\boldsymbol{\beta} = (\cos\theta, \sin\theta, 0)$.

5. Another example of a ruled surface is a **Möbius strip** (or Möbius band). A model can be created by taking a paper strip and giving it a half-twist (180°-twists), and then joining the ends of the strip together to form a loop.

The Möbius strip has several curious properties: it is a surface with **only one side and only one boundary**. To convince yourself of these facts, create your own Möbius strip and play with it (or go to Wikipedia and study the images there).

Another interesting property is that if you cut a Möbius strip along the center line, you will get one long strip with two full twists in it, not two separate strips. The resulting strip will have two sides and two boundaries. So, cutting created a second boundary. Continuing this construction you can deduce that a strip with an odd-number of half-twists will have only one surface and one boundary while a strip with an even-number of half-twists will have two surfaces and two boundaries.



Möbius strip

For more curious properties and alternative construction of Möbius strip, see Wikipedia.

A Möbius strip can be obtained as a ruled surface by considering $\boldsymbol{\alpha}$ to be a unit-circle in xy-plane $(\cos t, \sin t, 0)$. Through each point of $\boldsymbol{\alpha}$ pass a line segment of unit length with midpoint $\boldsymbol{\alpha}(t)$ in direction of $\boldsymbol{\beta}(t) = \sin\frac{t}{2}\boldsymbol{\alpha}(t) + \cos\frac{t}{2}(0, 0, 1)$. The ruled surface $\mathbf{x}(t, s) = \boldsymbol{\alpha}(t) + s\boldsymbol{\beta}(t)$ is a Möbius strip.

There are many applications of Möbius strip in science, technology and everyday life. For example, Möbius strips have been used as conveyor belts (that last longer because the entire surface area of the belt gets the same amount of wear), fabric computer printer and typewriter ribbons. Medals often have a neck ribbon configured as a Möbius strip that allows the ribbon to fit comfortably around the neck while the medal lies flat on the chest. Examples of Möbius strip can be encountered: in physics as compact resonators and as superconductors with high transition temperature; in chemistry as molecular knots with special characteristics (e.g. chirality); in music theory as dyads and other areas.

The First Fundamental Form

The first fundamental form describes the way of measuring the distances on a surface. An apparatus that enables one to measure the distances is called **metric**. This is why the first fundamental form is often referred to as the metric form.

Since the basis of the tangent plane $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ will play a major role in the definition of the metric form, we use the usual abbreviation and denote them by $\mathbf{x}_1$ and $\mathbf{x}_2$.

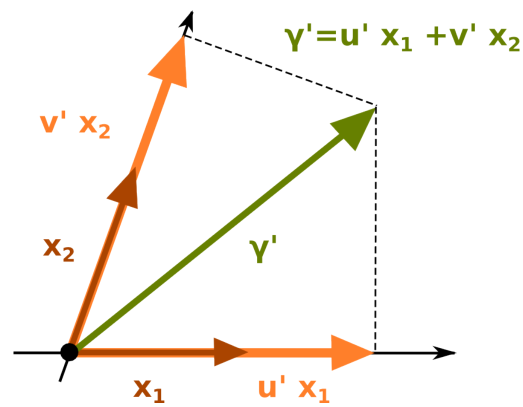
The condition $\mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0}$ guarantees that the tangent plane is not collapsed into a line or a point i.e. that it is a two-dimensional plane. It also implies that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ can be taken to be a **basis of the tangent plane.**

In particular, this means that the velocity vector of every curve on the surface can be represented via $\mathbf{x}_1$ and $\mathbf{x}_2$. Since the arc length of the curve can be found by integrating the length of the velocity vector, such **length will be computed by an integral involving $\mathbf{x}_1$ and $\mathbf{x}_2$**. This leads to the definition of the first fundamental form.

Let us start by considering the arc length of a curve $\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t))$ on a surface $\mathbf{x}$. The velocity vector $\boldsymbol{\gamma}'(t)$ is given by the chain rule $\frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt}$ which, using our new abbreviation, can be written as

$$\boldsymbol{\gamma}'(t) = \mathbf{x}_1 \frac{du}{dt} + \mathbf{x}_2 \frac{dv}{dt} = u'\mathbf{x}_1 + v'\mathbf{x}_2.$$

Thus, the velocity vector is a linear combination of the basis vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ with coefficients u' and v'.



The length on the curve is given by $L = \int_a^b |\boldsymbol{\gamma}'(t)| dt$. The square of length $|\boldsymbol{\gamma}'(t)|^2$ is equal to the dot product $\boldsymbol{\gamma}'(t) \cdot \boldsymbol{\gamma}'(t) = (u'\mathbf{x}_1 + v'\mathbf{x}_2) \cdot (u'\mathbf{x}_1 + v'\mathbf{x}_2)$, thus

$$|\boldsymbol{\gamma}'(t)|^2 = (u')^2\, \mathbf{x}_1 \cdot \mathbf{x}_1 + 2u'v'\, \mathbf{x}_1 \cdot \mathbf{x}_2 + (v')^2\, \mathbf{x}_2 \cdot \mathbf{x}_2.$$

Thus, the three dot products featured in this formula completely determine the arc length of any curve on the surface. To further abbreviate the notation, the dot products are denotes as follows

$$g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1, \quad g_{12} = \mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_1 = g_{21}, \quad g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2$$

and are called **the coefficients of the first fundamental form.**

The traditional notation $g_{11} = E, g_{12} = F$, and $g_{22} = G$ comes from Gauss. The more modern notation $g_{11}, g_{12} = g_{21}$, and g_{22} is convenient for representing the relevant dot products as a matrix $[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$.

Using this notation, $|\boldsymbol{\gamma}'(t)|^2 = g_{11}(u')^2 + 2g_{12}u'v' + g_{22}(v')^2$ so that the length of the curve $\boldsymbol{\gamma}$ on the surface $\mathbf{x}$ is given by

$$L = \int_a^b |\boldsymbol{\gamma}'(t)| dt = \int_a^b \left(g_{11}(u')^2 + 2g_{12}u'v' + g_{22}(v')^2\right)^{1/2} dt = \int_a^b \left(g_{11}\, du^2 + 2g_{12}\, dudv + g_{22}\, dv^2\right)^{1/2}$$

The expression under the root, $g_{11}\,du^2 + 2g_{12}\,dudv + g_{22}\,dv^2$ is called the **first fundamental form**.

Examples.

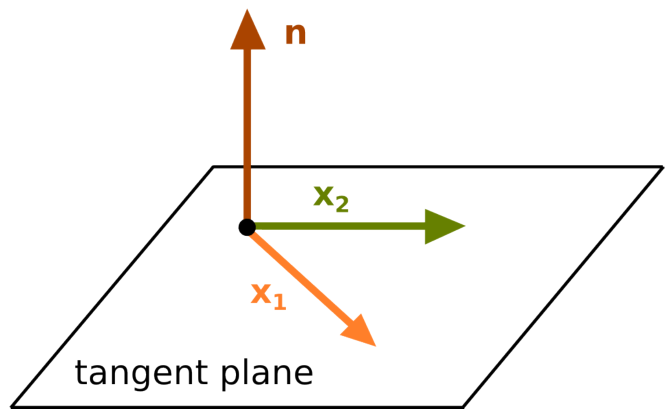
1. The the xy-plane $\mathbf{x} = (x, y, 0)$ has the tangent plane equal to the plane itself. The basis vectors are $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial x} = (1, 0, 0)$ and $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial y} = (0, 1, 0)$. Thus, the first fundamental form is $g_{11} = 1$, $g_{12} = 0$ and $g_{22} = 1$. The matrix g is the identity matrix reflecting the fact that the metric on xy-plane is the usual, standard metric. The arc length formula is the formula from Calculus 2 that computes the arc length of parametric curve $(x(t), y(t))$ as $L = \int_a^b \sqrt{(x')^2 + (y')^2}dt$.

2. If a surface is given by the formula $z = z(x, y)$, the basis vectors are $\mathbf{x}_1 = (1, 0, z_1)$ and $\mathbf{x}_2 = (0, 1, z_2)$ where z_1 and z_2 denote the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Thus, the first fundamental form is $g_{11} = 1 + z_1^2$, $g_{12} = z_1 z_2$ and $g_{22} = 1 + z_2^2$.

The nonzero vector $\mathbf{x}_1 \times \mathbf{x}_2$ is perpendicular to the tangent plane. Thus, **the unit normal vector** of the tangent plane is given by

$$\boxed{\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}}$$

This vector should not be confused with the normal vector $\mathbf{N}$ of a curve on a surface. In fact, the vectors $\mathbf{n}$ and $\mathbf{N}$ may have different direction.



For example, let $\boldsymbol{\gamma}$ be a circle obtained by intersection of a sphere and a plane that does not contain the center of the sphere. In this case, the radius of $\boldsymbol{\gamma}$ is less that the radius of the sphere and the center of $\boldsymbol{\gamma}$ is different than the center of the sphere. If P is a point on $\boldsymbol{\gamma}$ then the direction of $\mathbf{n}$ is determined by the line connecting P and the center of the sphere and the direction of $\mathbf{N}$ is determined by the line connecting P and the center of $\boldsymbol{\gamma}$.

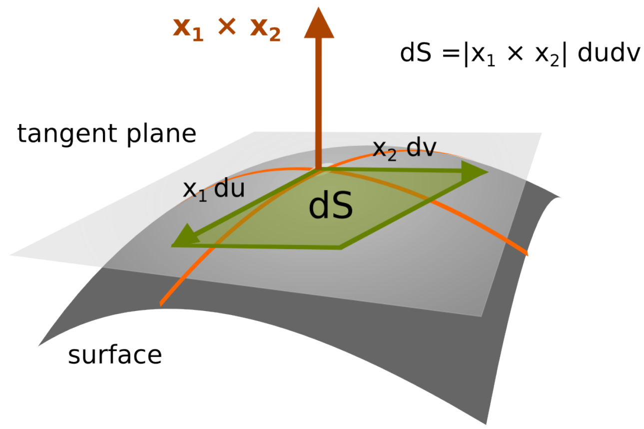
Using Lagrange identity $|\mathbf{x}_1 \times \mathbf{x}_2|^2 = (\mathbf{x}_1 \cdot \mathbf{x}_1)(\mathbf{x}_2 \cdot \mathbf{x}_2) - (\mathbf{x}_1 \cdot \mathbf{x}_2)^2$, we have that $|\mathbf{x}_1 \times \mathbf{x}_2|^2 = g_{11}g_{22} - g_{12}^2$, the determinant of the matrix $[g_{ij}]$. The determinant $g_{11}g_{22} - g_{12}^2$ is usually denoted by g. Thus,

$$|\mathbf{x}_1 \times \mathbf{x}_2|^2 = g \quad \text{ and } \quad \mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g}}.$$

The first fundamental form enables us to compute lengths and angles on a surface. Lengths can be computed by the formula $L = \int_a^b \left(g_{11}\,du^2 + 2g_{12}\,dudv + g_{22}\,dv^2\right)^{1/2}$. The angle between two curves on a surface can be defined as the angle between the two corresponding tangent vectors in the tangent plane. Recall that the angle α between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be found from the formula $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2| \cos\alpha$. Representing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ via the basis vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and expressing the dot product and the lengths via the coefficients g_{ij} we can obtain a formula computing the angle α in terms of the coefficients g_{ij}.

When we say that a certain quantity can be *measured intrinsically* we mean that it can be computed by measuring distances and angles on a surface without using any references to exterior space or the particular embedding. In particular, if certain quantity can be expressed solely in terms of the coefficients of the first fundamental form, it is an intrinsic quantity. Thus, to show Theorema Egregium, it is sufficient to show that the Gaussian curvature K can be computed solely using the coefficients of the first fundamental form.

Besides enabling us to compute lengths and angles on a surface, the first fundamental form also enables us to compute the surface areas. Note that the total area of region D on the surface can be computed by adding up all the areas of "rectangular" regions (i.e. approximately parallelogram shaped pieces on the surface) determined by the intersections of the u and v-curves.



Since the length of the cross product $|\mathbf{x}_1 \times \mathbf{x}_2|$ determines the area of parallelogram determined by $\mathbf{x}_1$ and $\mathbf{x}_2$, the area of one such "rectangular" region is given by

$$dS = |\mathbf{x}_1 \times \mathbf{x}_2|\, dudv = \sqrt{g}\, dudv$$

The total surface area of the surface $\mathbf{x}(u, v)$ over region S to be

$$\boxed{\text{Surface area } = \int\int_S dS = \int\int_S |\mathbf{x}_1 \times \mathbf{x}_2|\, dudv}$$

Recall that this is the familiar formula from Calculus 3 that calculates the surface area of a parametric surface.

Practice Problems.

1. Compute the first fundamental form, the determinant g of the matrix $[g_{ij}]$ and the unit normal vector for the following surfaces.

   (a) Surface given by $z = z(x, y)$.

   (b) Sphere given by $\mathbf{x} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, a\sin\phi)$.

   (c) Cylinder $x^2 + y^2 = 1$ (you can use parametrization $\mathbf{x} = (\cos\theta, \sin\theta, z)$).

   (d) Torus obtained by revolving a circle $(x - a)^2 + z^2 = b^2$ in xz-plane along the circle $x^2 + y^2 = a^2$ in xy-plane. Since the first circle can be parametrized by $x = a + b\cos\phi$, $z = b\sin\phi$ and a surface of revolution of a curve $x = f(u), z = g(u)$ in xz-plane about z-axis is given by the parametric equations $\mathbf{x} = (f(u)\cos\theta, f(u)\sin\theta, g(u))$, the torus can be parametrized as

   $$\mathbf{x} = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi).$$

2. Find the formula computing the surface area of a surface given by $z = z(x, y)$.

3. Find the area of the following surfaces.

   (a) The part of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1) and (0,1).

(b) The part of the surface $z = y^2 + x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Write down the parametric equations of the paraboloid and use them to find the surface area.

(c) The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Write down the parametric equations of the cone first. Then find the surface area using the parametric equations.

(d) The surface area of the cylinder $x^2 + z^2 = 4$ for $0 \le y \le 5$.

Solutions.

1. (a) We have found the basis vectors and the fundamental form to be $\mathbf{x}_1 = (1, 0, z_1)$, $\mathbf{x}_2 = (0, 1, z_2)$ and $g_{11} = 1 + z_1^2$, $g_{12} = z_1 z_2$ and $g_{22} = 1 + z_2^2$ (see the example above) where z_1 and z_2 denote the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. The determinant $g = (1+z_1^2)(1+z_2^2) - z_1^2 z_2^2 = 1 + z_1^2 + z_2^2 + z_1^2 z_2^2 - z_1^2 z_2^2 = 1 + z_1^2 + z_2^2$. The unit normal vector is given by $\mathbf{n} = \frac{1}{\sqrt{1+z_1^2+z_2^2}}(-z_1, -z_2, 1)$.

   (b) For the sphere, $\mathbf{x}_1 = (-a\sin\theta\cos\phi, a\cos\theta\cos\phi, 0)$, $\mathbf{x}_2 = (-a\cos\theta\sin\phi,\ -a\sin\theta\sin\phi,\ a\cos\phi)$. So, $g_{11} = a^2\cos^2\phi$, $g_{12} = 0$, $g_{22} = a^2$. Hence $g = a^4\cos^2\phi$. $\mathbf{n} = \frac{1}{a^2\cos\phi}(a^2\cos\theta\cos^2\phi,\ a^2\sin\theta\cos^2\phi,\ a^2\sin\phi\cos\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi) = \frac{1}{a}\mathbf{x}$.

   (c) For the cylinder, $\mathbf{x}_1 = (-\sin\theta, \cos\theta, 0)$, $\mathbf{x}_2 = (0, 0, 1)$. Thus, $g_{11} = 1$, $g_{12} = 0$ and $g_{22} = 1$. Hence $g = 1$ meaning that the lengths on the cylinder are the same as in a plane i.e. that the geometry of the cylinder locally is indistinguishable from the geometry of a plane (so the cylinder and a plane are locally isometric). The normal vector is $\mathbf{n} = (\cos\theta, \sin\theta, 0)$.

   (d) For the torus, $\mathbf{x}_1 = (-(a + b\cos\phi)\sin\theta, (a + b\cos\phi)\cos\theta, 0)$ and $\mathbf{x}_2 = (-b\sin\phi\cos\theta,\ -b\sin\phi\sin\theta,\ b\cos\phi)$. So, $g_{11} = (a + b\cos\phi)^2$, $g_{12} = 0$, $g_{22} = b^2$, $g = b^2(a + b\cos\phi)^2$, and $\mathbf{n} = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi)$.

2. Using problem 1, the surface are is computed by $A = \int\int_D \sqrt{1 + z_1^2 + z_2^2}\,dxdy$ which is the familiar formula for surface area from Calculus 3.

3. (a) 1.4 (b) Parametrization: $x = r\cos t$, $y = r\sin t$, $z = x^2 + y^2 = r^2$. The surface area is 30.85. (c) Parametrization: $x = r\cos t$, $y = r\sin t$, $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. The length of the cross product is $\sqrt{2}r$. The surface area is $5\pi\sqrt{2}$. (d) Parametrization: $x = 2\cos t$, $y = y$, $z = 2\sin t$. Bounds: $0 \le t \le 2\pi$, $0 \le y \le 5$. Length of the cross product is 2. Thus the double integral is $2\pi \cdot 5 \cdot 2 = 20\pi$.