Differential Geometry
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The Second Fundamental Form. Geodesics. The Curvature Tensor. The Fundamental Theorem of Surfaces. Manifolds

The Second Fundamental Form.

Consider a surface $\mathbf{x} = \mathbf{x}(u, v)$. Following the reasoning that $\mathbf{x}_1$ and $\mathbf{x}_2$ denote the derivatives $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ respectively, we denote the second derivatives

$$\frac{\partial^2 \mathbf{x}}{\partial u^2} \text{ by } \mathbf{x}_{11},\ \frac{\partial^2 \mathbf{x}}{\partial v \partial u} \text{ by } \mathbf{x}_{12},\ \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \text{ by } \mathbf{x}_{21}, \text{ and } \frac{\partial^2 \mathbf{x}}{\partial v^2} \text{ by } \mathbf{x}_{22}.$$

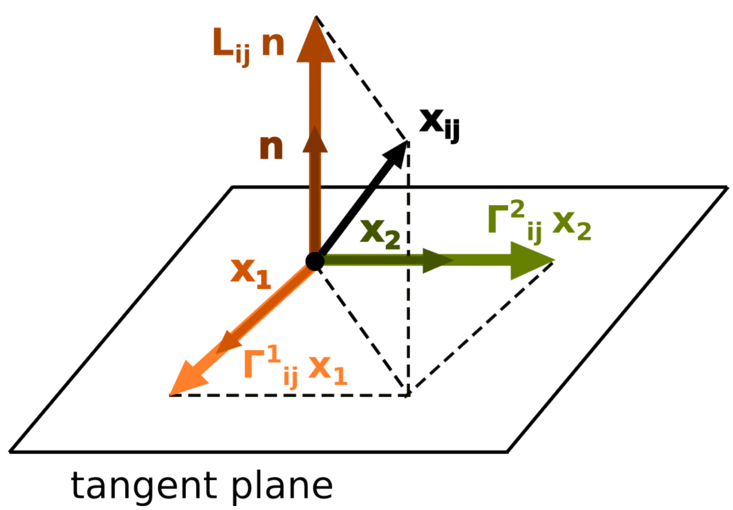
Using this notation, the second derivative of a curve $\boldsymbol{\gamma}$ on the surface $\mathbf{x}$ (obtained by differentiating $\boldsymbol{\gamma}'(t) = u'\mathbf{x}_1 + v'\mathbf{x}_2$ with respect to t) is given by

$$\boldsymbol{\gamma}'' = u''\mathbf{x}_1 + u'(u'\mathbf{x}_{11} + v'\mathbf{x}_{12}) + v''\mathbf{x}_2 + v'(u'\mathbf{x}_{21} + v'\mathbf{x}_{22}) = u''\mathbf{x}_1 + v''\mathbf{x}_2 + u'^2\mathbf{x}_{11} + u'v'\mathbf{x}_{12} + u'v'\mathbf{x}_{21} + v'^2\mathbf{x}_{22}.$$

The terms $u''\mathbf{x}_1 + v''\mathbf{x}_2$ are in the tangent plane (so this is the tangential component of $\boldsymbol{\gamma}''$).

The terms $\mathbf{x}_{ij}$, $i, j = 1, 2$ can be represented as a linear combination of tangential and normal component. Each of the vectors $\mathbf{x}_{ij}$ can be represented as a combination of the tangent component (which itself is a combination of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$) and the normal component (which is a multiple of the unit normal vector $\mathbf{n}$). Let Γ^1_{ij} and Γ^2_{ij} denote the coefficients of the tangent component and L_{ij} denote the coefficient with $\mathbf{n}$ of vector $\mathbf{x}_{ij}$. Thus,

$$\mathbf{x}_{ij} = \Gamma^1_{ij}\mathbf{x}_1 + \Gamma^2_{ij}\mathbf{x}_2 + L_{ij}\mathbf{n} = \sum_k \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}.$$



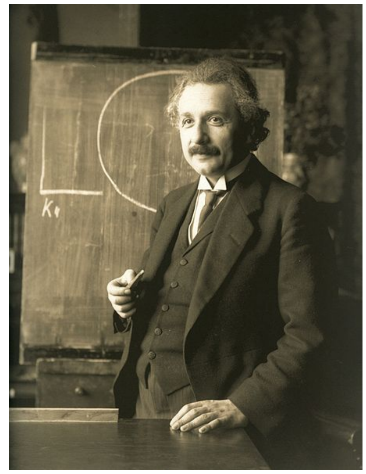
The formula above is called the **Gauss formula**.

The coefficients Γ^k_{ij} where $i, j, k = 1, 2$ are called **Christoffel symbols** and the coefficients L_{ij}, $i, j = 1, 2$ are called **the coefficients of the second fundamental form**.

Einstein notation and tensors. The term "Einstein notation" refers to the certain summation convention that appears often in differential geometry and its many applications. Consider a formula can be written in terms of a sum over an index that appears in subscript of one and superscript of the other variable. For example, $\mathbf{x}_{ij} = \sum_k \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$. In cases like this *the summation symbol is omitted*. Thus, the **Gauss formula** for $\mathbf{x}_{ij}$ in Einstein notation is written simply as

$$\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}.$$

The important benefit of the use of Einstein notation can be seen when considering n-dimensional manifolds – all the formulas we consider for surfaces generalize to formulas for n-dimensional manifolds. For example, the formula $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ remains true except that the indices i, j take integer values ranging from 1 to n not just values 1 and 2.



If we consider the scalar components in certain formulas as arrays of scalar functions, we arrive to the concept of a

tensor.

For example, a 2×2 matrix with entries g_{ij} is considered to be a tensor of rank 2. This matrix is referred to as the **metric tensor.** The scalar functions Γ^k_{ij} are considered to be the components of a tensor Γ of rank 3 (or type (2,1)). The Einstein notation is crucial for simplification of some complicated tensors.

Another good example of the use of Einstein notation is the matrix multiplication *(students who did not take Linear Algebra can skip this example and the next several paragraphs that relate to matrices).* If A is an $n \times m$ matrix and $\mathbf{v}$ is a $m \times 1$ (column) vector, the product $A\mathbf{v}$ will be a $n \times 1$ column vector. If we denote the elements of A by a^i_j where $i = 1, \ldots, n$, $j = 1, \ldots, m$ and x^j denote the entries of vector $\mathbf{x}$, then the entries of the product $A\mathbf{x}$ are given by the sum $\sum_j a^i_j x^j$ that can be denoted by $a^i_j x^j$ using Einstein notation.

Note also that the entries of a column vector are denoted with indices in superscript and the entries of row vectors with indices in subscript. This convention agrees with the fact that the entries of the column vector $a^i_j x^j$ depend just on the superscript i.

Another useful and frequently considered tensor is the Kronecker delta symbol. Recall that the identity matrix I is a matrix with the ij-th entry 1 if $i = j$ and 0 otherwise. Denote these entries by δ^i_j. Thus,

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In this notation, the equation $I\mathbf{x} = \mathbf{x}$ can be written as $\delta^i_j x^j = x^i$.

Recall that the inverse of a matrix A is the matrix A^{-1} with the property that the products AA^{-1} and $A^{-1}A$ are both equal to the identity matrix I. If a_{ij} denote the elements of the matrix A, a^{ij} denote the elements of the inverse matrix A^{ij}, the ij-th element of the product $A^{-1}A$ in Einstein notation is given by $a^{ik}a_{kj}$. Thus $a^{ik}a_{kj} = \delta^i_j$.

In particular, let g^{ij} denote the entries of the inverse matrix of $[g_{ij}]$ whose entries are the coefficients of the first fundamental form. The fact that the matrix and its inverse multiply to the identity gives us the following formulas (all given in Einstein notation).

$$g_{ik}g^{kj} = \delta^j_i \text{ and } g^{ik}g_{kj} = \delta^i_j.$$

The coefficients g^{ij} of the inverse matrix are given by the formulas

$$g^{11} = \frac{g_{22}}{g}, \quad g^{12} = \frac{-g_{12}}{g}, \quad \text{and } g^{22} = \frac{g_{11}}{g}$$

where g is the determinant of the matrix $[g_{ij}]$.

Computing the second fundamental form and the Christoffel symbols. The formula computing the Christoffel symbols can be obtained by multiplying the equation $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ by $\mathbf{x}_l$ where $k = 1, 2$. Since $\mathbf{n}\cdot\mathbf{x}_l = 0$, and $\mathbf{x}_k \cdot \mathbf{x}_l = g_{kl}$, we obtain that

$$\mathbf{x}_{ij}\cdot\mathbf{x}_l = \Gamma^k_{ij} g_{kl}$$

To solve for Γ^k_{ij}, we have to get rid of the terms g_{kl} from the left side. This can be done by using the inverse matrix g^{ls}.

$$(\mathbf{x}_{ij}\cdot\mathbf{x}_l)g^{ls} = \Gamma^k_{ij} g_{kl} g^{ls} = \Gamma^k_{ij}\delta^s_k = \Gamma^s_{ij}$$

Thus, we obtain that the Christoffel symbols can be computed by the formula

$$\boxed{\Gamma^k_{ij} = (\mathbf{x}_{ij}\cdot\mathbf{x}_l)g^{lk}.}$$

To compute the coefficients of the second fundamental form, multiply the equation $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ by $\mathbf{n}$. Since $\mathbf{x}_l\cdot\mathbf{n} = 0$, we have that $\mathbf{x}_{ij}\cdot\mathbf{n} = L_{ij}\mathbf{n}\cdot\mathbf{n} = L_{ij}$. Thus,

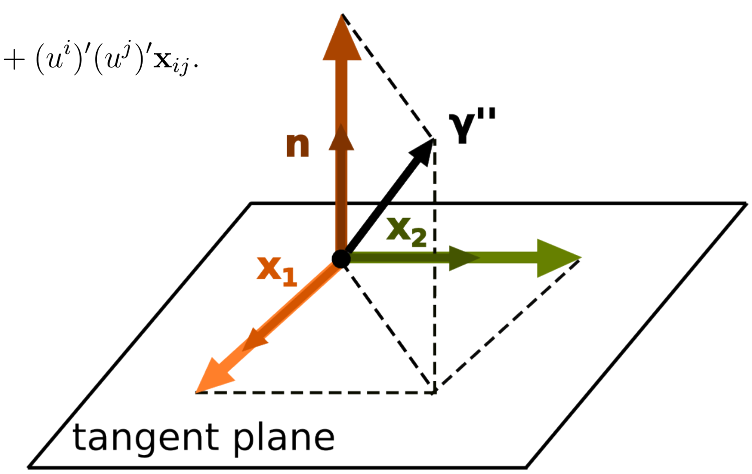
$$\boxed{L_{ij} = \mathbf{x}_{ij}\cdot\mathbf{n} = \mathbf{x}_{ij}\cdot\frac{\mathbf{x}_1\times\mathbf{x}_2}{|\mathbf{x}_1\times\mathbf{x}_2|}.}$$

The second fundamental form. Recall that we obtained the formula for the second derivative $\boldsymbol{\gamma}''$ to be

$$\boldsymbol{\gamma}'' = u''\mathbf{x}_1 + v''\mathbf{x}_2 + u'^2\mathbf{x}_{11} + u'v'\mathbf{x}_{12} + u'v'\mathbf{x}_{21} + v'^2\mathbf{x}_{22}.$$

To be able to use Einstein notation, let us denote u by u^1 and v by u^2. Thus the part $u''\mathbf{x}_1 + v''\mathbf{x}_2$ can be written as $(u^i)''\mathbf{x}_i$ and the part $u'v'\mathbf{x}_{12} + u'v'\mathbf{x}_{21} + v'^2\mathbf{x}_{22}$ as $(u^i)'(u^j)'\mathbf{x}_{ij}$. This gives us the short version of the formula above

$$\boldsymbol{\gamma}'' = (u^i)''\mathbf{x}_i + (u^i)'(u^j)'\mathbf{x}_{ij}.$$



Substituting the Gauss formula $\mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}$ in the formula, we obtain that

$$\boldsymbol{\gamma}'' = (u^i)''\mathbf{x}_i + (u^i)'(u^j)'(\Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}) =$$

$$((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k + (u^i)'(u^j)'L_{ij}\mathbf{n}.$$

The part $((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k$ is the **tangential component** and it is in the tangent plane. The part $(u^i)'(u^j)'L_{ij}\mathbf{n}$ is the **normal component** and it is orthogonal to the tangent plane.

The coefficients $(u^i)'(u^j)'L_{ij}$ of the normal component are the **second fundamental form**. While the first fundamental form determines the intrinsic geometry of the surface, the second fundamental form reflects the way how the surface embeds in the surrounding space and how it curves relative to that space. Thus, the second fundamental form reflects the **extrinsic geometry of the surface**.

Practice Problems. Find the coefficients of the second fundamental form for the following surfaces.

1. $z = f(x, y)$

2. Sphere of radius a parametrized by geographic coordinates.

3. Cylinder of radius a. (you can use parametrization $(a\cos t, a\sin t, z)$).

Solutions.

1. Use x, y as parameters and shorten the notation by using z_1 for z_x, z_2 for z_y, and $z_{11} = z_{xx}$, $z_{12} = z_{21} = z_{xy}$ and $z_{22} = z_{yy}$. We have $\mathbf{x}_1 = (1, 0, z_1)$ and $\mathbf{x}_2 = (0, 1, z_1)$. $g_{11} = 1+z_1^2$, $g_{12} = z_1 z_2$ and $g_{22} = 1 + z_2^2$. Thus, $g = 1 + z_1^2 + z_2^2$, and $\mathbf{n} = \frac{1}{\sqrt{g}}(-z_1, -z_2, 1)$. Also, $\mathbf{x}_{11} = (0, 0, z_{11})$, $\mathbf{x}_{12} = (0, 0, z_{12})$, $\mathbf{x}_{22} = (0, 0, z_{22})$. Then calculate that $L_{ij} = \frac{z_{ij}}{\sqrt{g}}$.

2. Compute that $L_{11} = -a\cos^2\phi$, $L_{12} = 0$ and $L_{22} = -a$.

3. $\mathbf{x}_1 = (-a\sin t, a\cos t, 0)$, $\mathbf{x}_2 = (0, 0, 1)$. Thus, $g_{11} = a^2$, $g_{12} = 0$ and $g_{22} = 1$. Hence $g = a^2$, $\mathbf{n} = (\cos t, \sin t, 0)$, $\mathbf{x}_{11} = (-a\cos t, -a\sin t, 0)$ and $\mathbf{x}_{12} = \mathbf{x}_{22} = 0$. Thus, $L_{11} = -a$, and $L_{12} = L_{22} = 0$.

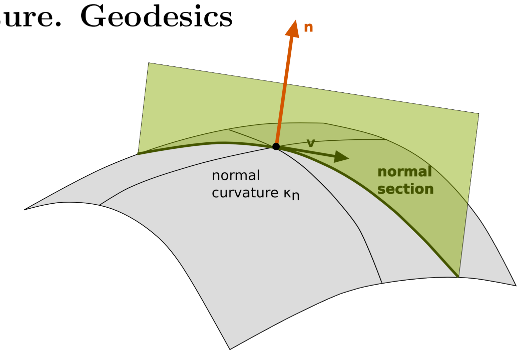
Normal and Geodesic curvature. Geodesics

The curvature of a curve $\boldsymbol{\gamma}$ on a surface is impacted by two factors.

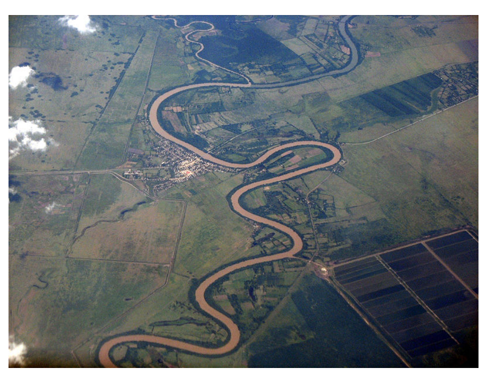
1. **External curvature of the surface**. If a surface itself is curved relative to the surrounding space in which it embeds, then a curve on this surface will be forced to bend as well. The level of this bending is measured by the **normal curvature** κ_n.

   For example, the curving of any curve in a normal section of a surface comes just from curving of the surface itself.

2. **Curvature of the curve relative to the surface.** Consider a curve "meandering" in a plane. The curvature of this curve comes only from the "meandering, not from any exterior curving of the plane since the plane is flat. This level of bending is measured by the **geodesic curvature** κ_g.



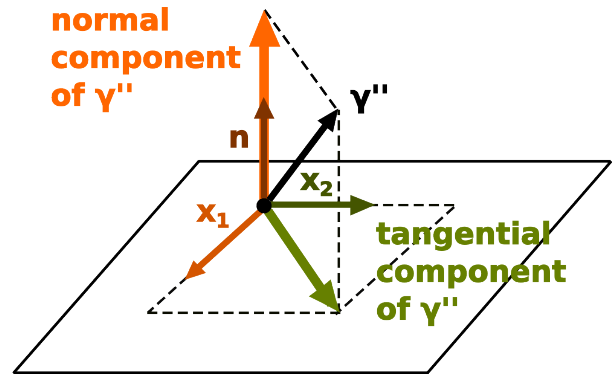
Example with $\kappa_n \neq 0, \kappa_g = 0$



Example with $\kappa_n = 0, \kappa_g \neq 0$

In this case, a meandering curve in a plane has the normal curvature κ_n equal to zero and the geodesic curvature κ_g nonzero.

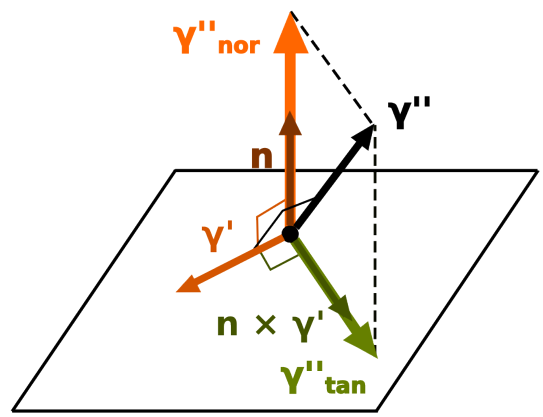
We now examine more closely meaning and computation of the two curvatures. Consider a curve $\boldsymbol{\gamma}$ on a surface and assume that it is *parametrized by the arc length.* In this case, the length of the second derivative computes the curvature $\kappa = |\boldsymbol{\gamma}''|$.



Recall that the vector $\boldsymbol{\gamma}''$ can be decomposed into the sum of the tangential and the normal component $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}''_{tan} + \boldsymbol{\gamma}''_{nor} = ((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k + (u^i)'(u^j)' L_{ij}\mathbf{n}$.

Up to the sign, the length of the tangential component $\boldsymbol{\gamma}''_{tan}$ determines the geodesic curvature κ_g and the length of the normal component $\boldsymbol{\gamma}''_{nor}$ determines the normal curvature κ_n. Dotting the last formula by $\mathbf{n}$ produces the formula which calculates κ_n.

$$\kappa_n = \boldsymbol{\gamma}'' \cdot \mathbf{n} = (u^i)'(u^j)' L_{ij}.$$



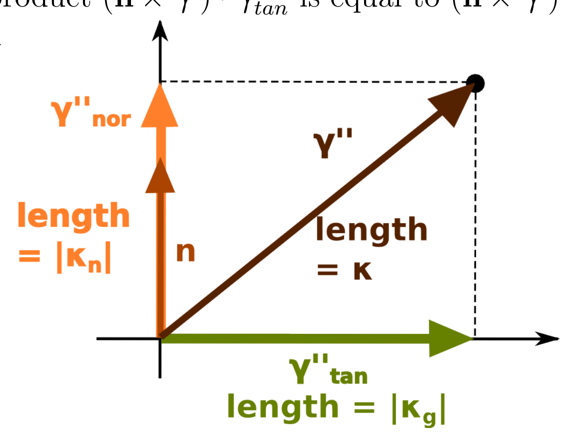
A formula for computing κ_g can be obtained by expressing $\boldsymbol{\gamma}''_{tan}$ in terms of different basis of the tangent plane, not $\mathbf{x}_1$ and $\mathbf{x}_2$. To do this, start by noting that $\boldsymbol{\gamma}''_{nor} \cdot \boldsymbol{\gamma}' = 0$ since $\boldsymbol{\gamma}'$ is in the tangent plane. Also, $\boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}' = 0$ since $\boldsymbol{\gamma}'$ is a vector of constant length. Note that here we are using the same argument we have utilized multiple times in the course, in particular in section on curves. Thus

$$0 = \boldsymbol{\gamma}'' \cdot \boldsymbol{\gamma}' = (\boldsymbol{\gamma}''_{tan} + \boldsymbol{\gamma}''_{nor}) \cdot \boldsymbol{\gamma}' = \boldsymbol{\gamma}''_{tan} \cdot \boldsymbol{\gamma}' + \boldsymbol{\gamma}''_{nor} \cdot \boldsymbol{\gamma}' = \boldsymbol{\gamma}''_{tan} \cdot \boldsymbol{\gamma}' + 0 = \boldsymbol{\gamma}''_{tan} \cdot \boldsymbol{\gamma}'.$$

So, $\boldsymbol{\gamma}''_{tan}$ is orthogonal to $\boldsymbol{\gamma}'$ as well. Thus, $\boldsymbol{\gamma}''_{tan}$ is orthogonal to both $\boldsymbol{\gamma}'$ and $\mathbf{n}$ and thus it is *colinear with* $\mathbf{n} \times \boldsymbol{\gamma}'$. The **geodesic curvature** κ_g is the proportionality constant

$$\boldsymbol{\gamma}''_{tan} = \kappa_g(\mathbf{n} \times \boldsymbol{\gamma}')$$

Thus, the length $|\boldsymbol{\gamma}''_{tan}|$ is $\pm\kappa_g$. Dotting the above identity by $\mathbf{n} \times \boldsymbol{\gamma}'$, we obtain $(\mathbf{n} \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}''_{tan} = \kappa_g$. But since $\mathbf{n} \times \boldsymbol{\gamma}'$ is perpendicular to $\boldsymbol{\gamma}''_{nor}$, the mixed product $(\mathbf{n} \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}''_{tan}$ is equal to $(\mathbf{n} \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}''$. Thus $\kappa_g = (\mathbf{n} \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}''$ or, using the bracket notation



$$\boxed{\kappa_g = (\mathbf{n} \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}'' = [\mathbf{n}, \boldsymbol{\gamma}', \boldsymbol{\gamma}''] = [\mathbf{n}, \mathbf{T}, \mathbf{T}'].}$$

The curvatures $\kappa = |\boldsymbol{\gamma}''|, \kappa_g = \pm|\boldsymbol{\gamma}''_{tan}|$, and $\kappa_n = \pm|\boldsymbol{\gamma}''_{nor}|$ are related by the formula

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

which can be seen from the figure on the right.

Geodesics. A curve $\boldsymbol{\gamma}$ on a surface is said to be **a geodesic** if $\kappa_g = 0$ at every point of $\boldsymbol{\gamma}$. The following conditions are equivalent.

1. $\boldsymbol{\gamma}$ is a geodesic.	2. $[\mathbf{n}, \mathbf{T}, \mathbf{T}'] = 0$.
3. $\boldsymbol{\gamma}''_{tan} = 0$ at every point of $\boldsymbol{\gamma}$.	4. $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}''_{nor}$ at every point of $\boldsymbol{\gamma}$.
5. $(u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)' = 0$ for $k = 1, 2$.	6. $\kappa = \pm\kappa_n$ at every point of $\boldsymbol{\gamma}$.
7. $\mathbf{N}$ is colinear with $\mathbf{n}$ (i.e. $\mathbf{N} = \pm\mathbf{n}$).	

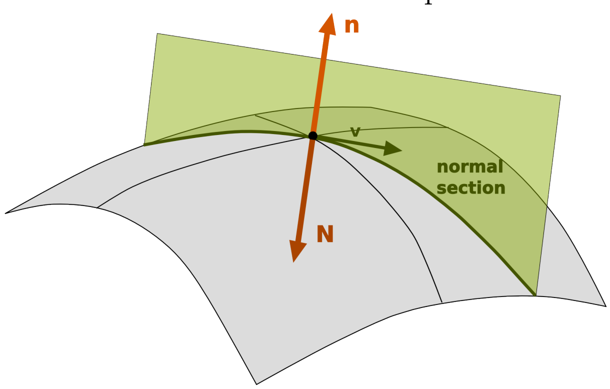
The conditions 1 and 2 are equivalent since $\kappa_g = [\mathbf{n}, \mathbf{T}, \mathbf{T}']$. The conditions 3 and 4 are clearly equivalent. The conditions 3 and 5 are equivalent since $\boldsymbol{\gamma}''_{tan} = ((u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)')\mathbf{x}_k$. The conditions 1 and 3 are equivalent since $\boldsymbol{\gamma}''_{tan} = 0 \Leftrightarrow \kappa_g = |\boldsymbol{\gamma}''_{tan}| = 0$.

To see that the conditions 1 and 6 are equivalent, recall the formula $\kappa_n^2 = \kappa_g^2 + \kappa_n^2$. Thus, if $\kappa_g = 0$ then $\kappa^2 = \kappa_n^2 \Rightarrow \kappa = \pm\kappa_n$. Conversely, if $\kappa = \pm\kappa_n$, then $\kappa^2 = \kappa_n^2 \Rightarrow \kappa_g^2 = 0 \Rightarrow \kappa_g = 0$.

Finally, to show that 1 and 7 are equivalent, recall that $\boldsymbol{\gamma}'' = \mathbf{T}' = \kappa\mathbf{N}$ if $\boldsymbol{\gamma}$ is parametrized by arc length. Assuming that $\boldsymbol{\gamma}$ is a geodesic, we have that $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}''_{nor} = \kappa_n\mathbf{n}$. Thus, $\kappa\mathbf{N} = \kappa_n\mathbf{n}$ and so the vectors $\mathbf{N}$ and $\mathbf{n}$ are colinear, in particular $\mathbf{N} = \pm\mathbf{n}$ since they both have unit length. Conversely, if $\mathbf{N}$ and $\mathbf{n}$ are colinear, then $\boldsymbol{\gamma}''$ (always colinear with $\mathbf{N}$ if unit-length parametrization is used) is colinear with $\mathbf{n}$ as well. So $\boldsymbol{\gamma}'' = \boldsymbol{\gamma}_{nor}$ and so condition 4 holds. Since we showed that 1 and 4 are equivalent, 1 holds as well. This concludes the proof that all seven conditions are equivalent.

Examples.

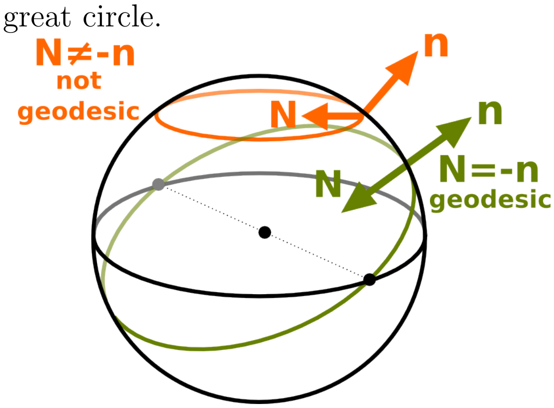
1. If $\boldsymbol{\gamma}$ is the normal section in the direction of a vector $\mathbf{v}$ in the tangent plane (intersection of the surface with a plane orthogonal to the tangent plane), then the normal vector $\mathbf{N}$ has the same direction as the unit normal vector $\mathbf{n}$ and so $\mathbf{N} = \pm\mathbf{n}$ (the sign is positive if the acceleration vector has the same direction as $\mathbf{n}$). So, **every normal section is a geodesic.**



2. A great circle on a sphere is the normal section and so, it is a geodesic. Having two points on a sphere which are not antipodal (i.e. exactly opposite to one another with respect to the center), there is a great circle on which the two points lie. Thus, the "straightest possible" curve on a sphere that connects any two points is a great circle.

   Thus, κ_g of a great circle is 0 and its curvature κ comes just from the normal curvature κ_n (equal to $\frac{1}{a}$ if the radius is a).

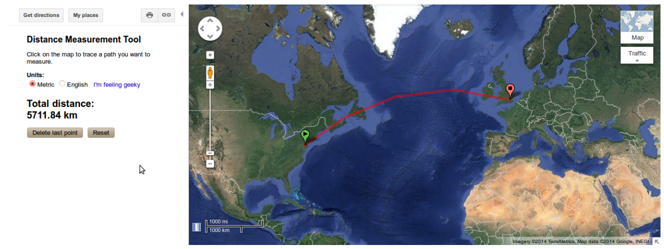
   Any circle on a sphere which is not "great" (i.e. whose center does not coincide with a center of the sphere and the radius is smaller than a) is not a geodesic. Any such "non-great" circle is an example of a curve on a surface whose normal vector $\mathbf{N}$ is not colinear with the normal vector of the sphere $\mathbf{n}$.



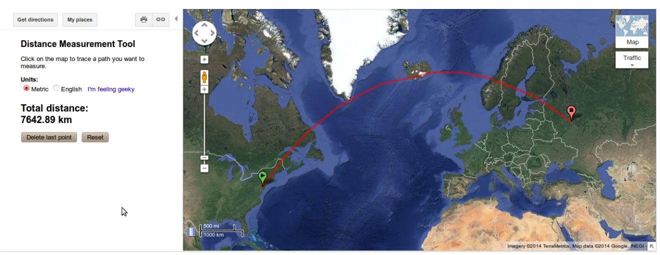
Just great circles are geodesics

Condition $(u^k)'' + \Gamma_{ij}^k (u^i)'(u^j)' = 0$ for $k = 1, 2$ (condition in the list on the previous page) can be seen as the set of differential equations (system of two differential equations of second order) whose solutions compute geodesics on a surface. Note that the differential equations depend just on the Christoffel symbols. These differential equations provide a tool for explicitly obtaining the formulas of geodesics on a surface. This tool is frequently used in everyday life, for example when determining the shortest flight route for an airplane.

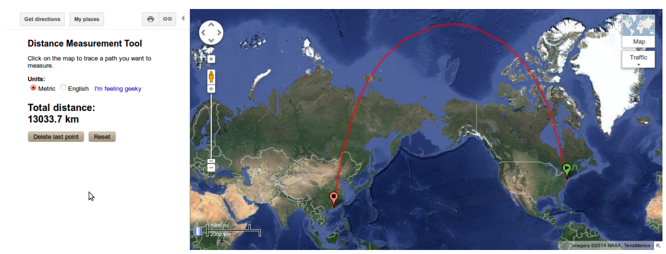
Consider, for example, air traffic routes from Philadelphia to London, Moscow and Hong Kong represented below. Each city being further from Philadelphia the previous one, makes the geodesic distance appear more curved when represented on the flat plane. Still, all three routes are determined as geodesics – as intersections of great circles on Earth which contain Philadelphia and the destination city.



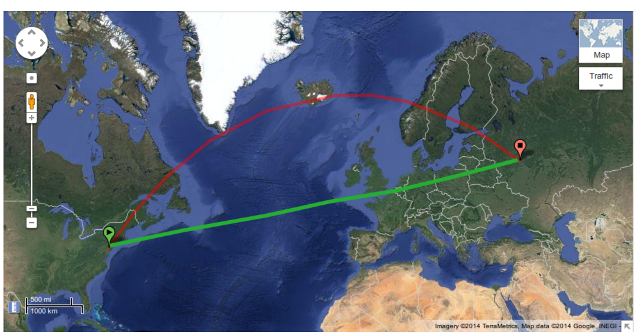
Philadelphia to London



Philadelphia to Moscow



Philadelphia to Hong Kong



Philadelphia to Moscow
via geodesic and
non-geodesics routes

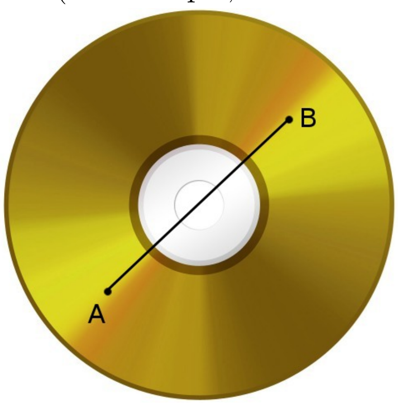
Since a geodesic curves solely because of curving of the surface, a geodesic has the role of a straight line on a surface. Moreover, geodesics have the following properties of straight lines.

1. If a curve $\boldsymbol{\gamma}(s)$ for $a \leq s \leq b$ is the shortest route on the surface that connects the points $\boldsymbol{\gamma}(a)$ and $\boldsymbol{\gamma}(b)$, then $\boldsymbol{\gamma}$ is a geodesic. This claim can be shown by computing the derivative of the arc length and setting it equal to zero. From this equation, the condition (5) follows and so $\boldsymbol{\gamma}$ is geodesic.

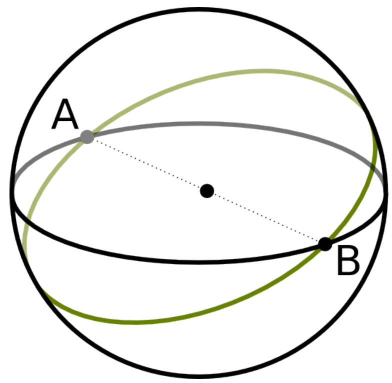
Note that the converse does not have to hold – if a curve is geodesic, it may not give the shortest route between its two points. For example, a north pole and any other point on a sphere but the south pole, determine two geodesics connecting them, just one of which will be the shortest route.

2. Every point P on a surface and a vector $\mathbf{v}$ in the tangent plane uniquely determine a geodesic $\boldsymbol{\gamma}$ with $\boldsymbol{\gamma}(0) = P$ and $\boldsymbol{\gamma}'(0) = \mathbf{v}$.

   As opposed to the straight lines, a geodesic connecting two points does not have to exist. For example, consider the xy-plane without the origin. Then there is no geodesic connecting (1,0) and (-1,0). Also, there can be infinitely many geodesics connecting two given points on a surface (for example, take north and south poles on a sphere).



Two points do not determine a “line”



There are many “lines” passing two points

Next, we show that the geodesic curvature can be computed intrinsically. Start by differentiating the equation $g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$ with respect to u^k. Get

$$\frac{\partial g_{ij}}{\partial u^k} = \mathbf{x}_{ik} \cdot \mathbf{x}_j + \mathbf{x}_i \cdot \mathbf{x}_{jk}$$

In similar manner, we obtain

$$\frac{\partial g_{ik}}{\partial u^j} = \mathbf{x}_{ij} \cdot \mathbf{x}_k + \mathbf{x}_i \cdot \mathbf{x}_{kj} \text{ and } \frac{\partial g_{jk}}{\partial u^i} = \mathbf{x}_{ji} \cdot \mathbf{x}_k + \mathbf{x}_j \cdot \mathbf{x}_{ki}$$

Note that the second equation can be obtained from the first by permuting the indices j and k and the third equation can be obtained from the second by permuting the indices i and j. This is called *cyclic permutation of indices.*

At this point, we require the second partial derivatives to be continuous as well. This condition will guarantee that the partial derivatives $\mathbf{x}_{ij}$ and $\mathbf{x}_{ji}$ are equal. In this case, adding the second and third equation and subtracting the first gives us

$$\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} = \mathbf{x}_{ij} \cdot \mathbf{x}_k + \underline{\mathbf{x}_i \cdot \mathbf{x}_{kj}} + \mathbf{x}_{ji} \cdot \mathbf{x}_k + \underline{\mathbf{x}_j \cdot \mathbf{x}_{ki}} - \underline{\mathbf{x}_{ik} \cdot \mathbf{x}_j} - \underline{\mathbf{x}_i \cdot \mathbf{x}_{jk}} = 2\mathbf{x}_{ij} \cdot \mathbf{x}_k$$

Thus,

$$\boxed{\Gamma^k_{ij} = (\mathbf{x}_{ij} \cdot \mathbf{x}_l) g^{lk} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}.}$$

This shows that the Christoffel symbols Γ^k_{ij} can be computed just in terms of the metric coefficients g_{ij} that can be determined by measurements within the surface. This proves the following theorem.

Theorem. The geodesic curvature is intrinsic.

Examples.

(1) Geodesics in a plane. With an appropriate choice of coordinates we may assume that this is the xy-plane, $z=0$ thus $\mathbf{x}=(x,y,0)$ (thus $u=x$ and $v=y$). The coefficient of the first fundamental form are $g_{11}=g_{22}=1$ and $g_{12}=0$. The Christoffel symbols vanish and thus the geodesics are given by equations $x''=0$ and $y''=0$. These equations have solutions

$$x = as+b \text{ and } y = cs+d$$

which represent parametric equations of a line. Hence, geodesics are straight lines.

(2) Geodesics on a cylinder. The cylinder $x^2+y^2=a^2$ can be parametrized as $\mathbf{x}=(a\cos t, a\sin t, z)$ so that $u=t$ and $v=z$. We have that $\mathbf{x}_1=(-a\sin t, a\cos t, 0)$, $\mathbf{x}_2=(0,0,1)$, $g_{11}=a^2$, $g_{12}=0=g^{12}$ and $g_{22}=1=g^{22}$. Hence $g=a^2$ and $g^{11}=\frac{1}{a^2}$ $\mathbf{n}=(\cos t,\sin t,0)$, $\mathbf{x}_{11}=(-a\cos t,-a\sin t,0)$ and $\mathbf{x}_{12}=\mathbf{x}_{22}=0$. Thus $\Gamma^k_{12}=\Gamma^k_{21}=\Gamma^k_{22}=0$ and Γ^k_{11} can also be computed to be 0. Thus, the equation of a geodesic is given by $t''=0$ and $z''=0$. These equations have solutions $t=as+b$ and $z=cs+d$ which are parametric equations of a line in tz-plane. This shows that a curve on a cylinder is geodesic if and only if it is a straight line in zt-plane.

In particular, both meridians and parallels are geodesics. The meridians are z-curves (parametrized by unit-length since $\mathbf{x}_2=(0,0,1)$ has unit length). Since $t=t_0$ is a constant on a z-curve, $t'=t''=0$ so the first equation holds. The second holds since $z'=\frac{dz}{dz}=1$ and so $z''=0$. Hence, both geodesic equations hold.

The parallels (or circles of latitude), are t-curves with $z=z_0$ a constant. They are parametrized by unit length for $t=\frac{s}{a}$. Thus, $t'=\frac{1}{a}$ and $t''=0$ and $z'=z''=0$ so both geodesic equations hold.

Another way to see that the circles of latitude $\boldsymbol{\gamma}=(a\cos t, a\sin t, z_0)=(a\cos\frac{s}{a}, a\sin\frac{s}{a}, z_0)$ are geodesics is to compute the second derivative (colinear with $\mathbf{N}$) and to note that it is a multiple of $\mathbf{n}$ (thus condition (4) holds). The first derivative is $\boldsymbol{\gamma}'=(-\sin\frac{s}{a},\cos\frac{s}{a},0)$ and the second is $\boldsymbol{\gamma}''=(-\frac{1}{a}\cos\frac{s}{a},-\frac{1}{a}\sin\frac{s}{a},0)$. The second derivative is a multiple of $\mathbf{n}=(\cos\frac{s}{a},\sin\frac{s}{a},0)$ ($\boldsymbol{\gamma}''=\frac{-1}{a}\mathbf{n}$) and so $\boldsymbol{\gamma}$ is a geodesic.

(3) Meridians of a cone are geodesics. Consider the cone obtained by revolving the line $(\frac{3}{5}s,\frac{4}{5}s)$ about z-axis so that the parametrization of the cone is $\mathbf{x}=\frac{1}{5}(3s\cos\theta, 3s\sin\theta, 4s)$. In this parametrization, the meridians are s-curves and parallels are θ-curves. Let us show that the meridians are geodesics.

Note that the meridians have the unit-speed parametrization since $\mathbf{x}_1=\frac{1}{5}(3\cos\theta,3\sin\theta,4)$ and $g_{11}=|x_1|^2=1$. $\mathbf{x}_2=\frac{1}{5}(-3s\sin\theta,3s\cos\theta,0)$ so that $g_{12}=0$ and $g_{22}=\frac{9}{25}s^2$. In addition $\mathbf{x}_{11}=(0,0,0)$, $\mathbf{x}_{12}=\frac{1}{5}(-3\sin\theta,3\cos\theta,0)$ and $\mathbf{x}_{22}=\frac{1}{5}(-3s\cos\theta,-3s\sin\theta,0)$. The inverse matrix of $[g_{ij}]$ is $\begin{bmatrix} 1 & 0 \\ 0 & \frac{25}{9s^2} \end{bmatrix}$. The Christoffel symbols can be computed as $\Gamma^k_{ij}=(\mathbf{x}_{ij}\cdot\mathbf{x}_l)g^{lk}$. Thus, $\Gamma^1_{11}=0$, $\Gamma^2_{11}=0$, $\Gamma^1_{12}=\Gamma^1_{21}=0$, $\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{s}$, $\Gamma^1_{22}=-\frac{9}{25}s$, $\Gamma^2_{22}=0$. Thus, the two equations of geodesics are $s''-\frac{9}{25}s(\theta')^2=0$ and $\theta''+\frac{2}{s}s'\theta'=0$ for a curve $\boldsymbol{\gamma}$ on the cone for which s and θ depend on a parameter t. These two equations represent the conditions for $\boldsymbol{\gamma}$ to be a geodesic.

The meridians are s-curves. Thus θ is constant so $\theta'=\theta''=0$ and $s'=1$, $s''=0$. So, both geodesic equations hold. This gives us that meridians on the cone are geodesics.

The Gauss Curvature. The Curvature Tensor. Theorema Egregium

Recall that the formula for the normal curvature is given by $\kappa_n = L_{ij}(u^i)'(u^j)'$. If the curve $\boldsymbol{\gamma}$ is not given by arc length parametrization, this formula becomes $\kappa_n = \frac{L_{ij}(u^i)'(u^j)'}{|\boldsymbol{\gamma}'|^2}$. Recall the formula for $|\boldsymbol{\gamma}'|^2$ from earlier section

$$|\boldsymbol{\gamma}'(t)|^2 = g_{11}((u^1)')^2 + 2g_{12}(u^1)'(u^2)' + g_{22}((u^2)')^2 \text{ in Einstein notation } = g_{ij}(u^i)'(u^j)'$$

Thus, the normal curvature can be computed as

$$\kappa_n = \frac{L_{ij}(u^i)'(u^j)'}{g_{kl}(u^k)'(u^l)'}$$

Differentiating this equation with respect to $(u^r)'$ for $r = 1, 2$, and setting derivatives to zero in order to get conditions for extreme values, we can obtain the conditions that $(L_{ij} - \kappa_n g_{ij})(u^j)' = 0$ for $i = 1, 2$. A nonzero vector $((u^1)', (u^2)')$ can be a solution of these equations just if the determinant of the system $|L_{ij} - \kappa_n g_{ij}|$ is zero.

This determinant is equal to $(L_{11} - \kappa_n g_{11})(L_{22} - \kappa_n g_{22}) - (L_{12} - \kappa_n g_{12})^2$. Substituting that determinant of $[g_{ij}]$ is g and denoting the determinant of $[L_{ij}]$ by L, we obtain the following quadratic equation in κ_n

$$g\kappa_n^2 - (L_{11}g_{22} + L_{22}g_{11} - 2L_{12}g_{12})\kappa_n + L = 0$$

The solutions of this quadratic equation are the principal curvatures κ_1 and κ_2. The Gauss curvature K is equal to the product $\kappa_1\kappa_2$ and from the above quadratic equation this product is equal to the quotient $\frac{L}{g}$ (recall that the product of the solutions x_1, and x_2 of a quadratic equation $ax^2 + bx + c$ is equal to $\frac{c}{a}$). Thus,

$$\boxed{K = \frac{L}{g}}$$

that is the Gauss curvature is the **quotient of the determinants of the coefficients of the second and the first fundamental forms**.

From the formula $(L_{ij} - \kappa_n g_{ij})(u^j)' = 0$ it follows that if $L_{12} = L_{21} = g_{12} = g_{21} = 0$, then the principal curvatures are given by $\frac{L_{11}}{g_{11}}$ and $\frac{L_{22}}{g_{22}}$ and the principal directions are $\mathbf{x}_1$ and $\mathbf{x}_2$. Conversely, if directions $\mathbf{x}_1$ and $\mathbf{x}_2$ are principal, then $L_{12} = L_{21} = g_{12} = g_{21} = 0$. Using this observation, we can conclude that the principal directions on a surface of revolution are determined by the meridian and the circle of latitude through every point.

Note that from the equation $(L_{ij} - \kappa_n g_{ij})(u^j)' = 0$ also follows that the principal curvatures are the eigenvalues of the operator determined by the first and the second fundamental form that can be expressed as

$$S = g^{-1}\begin{bmatrix} L_{11}g_{22} - L_{12}g_{12} & L_{12}g_{22} - L_{22}g_{12} \\ L_{12}g_{11} - L_{11}g_{12} & L_{22}g_{11} - L_{12}g_{12} \end{bmatrix}.$$

and is called the **shape operator**.

Examples.

1. We have computed the first and the second fundamental form of the sphere to be $[g_{ij}] = \begin{bmatrix} a^2\cos^2\phi & 0 \\ 0 & a^2 \end{bmatrix}$ and $[L_{ij}] = \begin{bmatrix} -a\cos^2\phi & 0 \\ 0 & -a \end{bmatrix}$. Thus $g = a^4\cos^2\phi$ and $L = a^2\cos^2\phi$. Hence

$$K = \frac{a^2\cos^2\phi}{a^4\cos^2\phi} = \frac{1}{a^2}.$$

2. We have also computed the first and the second fundamental form of the surface $z = f(x, y)$. $[g_{ij}] = \begin{bmatrix} 1+z_1^2 & z_1z_2 \\ z_1z_2 & 1+z_2^2 \end{bmatrix}$, $g = 1 + z_1^2 + z_2^2$, and $[L_{ij}] = \frac{1}{g}\begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}$. Thus

$$K = \frac{z_{11}z_{22} - z_{12}^2}{g^2} = \frac{z_{11}z_{22} - z_{12}^2}{(1 + z_1^2 + z_2^2)^2}.$$

Theorema Egregium. Recall that the informal statement of Theorema Egregium we presented before is that *the Gauss curvature can be calculated intrinsically.* In this section we present this result in the more formal way and prove it.

Recall the formula $K = \frac{L}{g}$. Thus, to prove the "Remarkable Theorem", we need to show that the determinant of the second fundamental form L is a function of the coefficients of the first fundamental form and their derivatives. The coefficient L_{ij} of the second fundamental form can be viewed as extrinsic because of the presence of the normal $\mathbf{n}$ in their definition. Theorema Egregium asserts that although the coefficients L_{ij} are extrinsic, their determinant L is intrinsic and can be computed solely via the first fundamental form.

We begin by introducing the Riemann curvature tensor (or Riemann-Christoffel curvature tensor) and showing three sets of equations known as Weingarten's, Gauss's and Codazzi-Mainardi equations.

The coefficients of the **Riemann curvature tensor** are defined via the Christoffel symbols by

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}$$

The geometric meaning of this tensor is not visible from this formula. It is possible to interpret this tensor geometrically, but the interpretation requires introduction of further concepts (covariant derivatives). So, we can think of it just as an aide to prove Theorema Egregium.

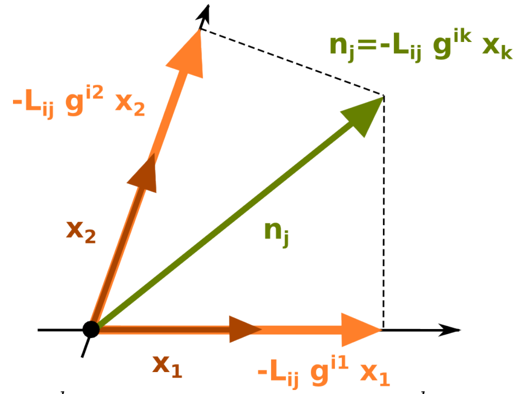
Proposition.

Weingarten's equations $\quad \mathbf{n}_j = -L_{ij}g^{ik}\mathbf{x}_k$

Gauss's equations $\quad R^l_{ijk} = L_{ik}L_{jp}g^{pl} - L_{ij}L_{kp}g^{pl}.$

Codazzi-Mainardi equations $\quad \frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \Gamma^l_{ik}L_{lj} - \Gamma^l_{ij}L_{lk}.$

Proof. Let us prove Weingarten's equations first. Since $\mathbf{n}\cdot\mathbf{n}=1$, $\mathbf{n}_j\cdot\mathbf{n}=0$ and so $\mathbf{n}_j$ is in tangent plane. Thus, it can be represented as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. Let a_j^l denote the coefficients of $\mathbf{n}_j$ with $\mathbf{x}_l$. Thus $\mathbf{n}_j = a_j^l\mathbf{x}_l$.



Differentiate the equation $\mathbf{n}\cdot\mathbf{x}_i=0$ with respect to u^j and obtain $\mathbf{n}_j\cdot\mathbf{x}_i+\mathbf{n}\cdot\mathbf{x}_{ij}=0$. Recall that $L_{ij}=\mathbf{n}\cdot\mathbf{x}_{ij}$.

Thus, $0=\mathbf{n}_j\cdot\mathbf{x}_i+L_{ij}=a_j^l\mathbf{x}_l\cdot\mathbf{x}_i+L_{ij}=a_j^lg_{li}+L_{ij}$ and so $a_j^lg_{li}=-L_{ij}$. To solve for a_j^l, multiply both sides by g^{ik} and recall that $g_{li}g^{ik}=\delta_l^k$. Thus we have $-L_{ij}g^{ik}=a_j^lg_{li}g^{ik}=a_j^l\delta_l^k=a_j^k$. This gives us

$$\mathbf{n}_j = a_j^k\mathbf{x}_k = -L_{ij}g^{ik}\mathbf{x}_k.$$

To prove the remaining two sets of equations, let us start by Gauss formulas for the second derivatives

$$\mathbf{x}_{ij} = \Gamma_{ij}^l\mathbf{x}_l + L_{ij}\mathbf{n}$$

Differentiate with respect to u^k and obtain

$$\begin{aligned}
\mathbf{x}_{ijk} &= \tfrac{\partial\Gamma_{ij}^l}{\partial u^k}\mathbf{x}_l + \Gamma_{ij}^l\mathbf{x}_{lk} + \tfrac{\partial L_{ij}}{\partial u^k}\mathbf{n} + L_{ij}\mathbf{n}_k\\
&= \tfrac{\partial\Gamma_{ij}^l}{\partial u^k}\mathbf{x}_l + \Gamma_{ij}^l(\Gamma_{lk}^p\mathbf{x}_p + L_{lk}\mathbf{n}) + \tfrac{\partial L_{ij}}{\partial u^k}\mathbf{n} - L_{ij}L_{pk}g^{pl}\mathbf{x}_l \text{ (sub Gauss and Wein. eqs)}\\
&= \tfrac{\partial\Gamma_{ij}^l}{\partial u^k}\mathbf{x}_l + \Gamma_{ij}^l\Gamma_{lk}^p\mathbf{x}_p - L_{ij}L_{pk}g^{pl}\mathbf{x}_l \ + \ \Gamma_{ij}^lL_{lk}\mathbf{n} + \tfrac{\partial L_{ij}}{\partial u^k}\mathbf{n} \text{ (regroup the terms)}\\
&= \tfrac{\partial\Gamma_{ij}^l}{\partial u^k}\mathbf{x}_l + \Gamma_{ij}^p\Gamma_{pk}^l\mathbf{x}_l - L_{ij}L_{pk}g^{pl}\mathbf{x}_l \ + \ \Gamma_{ij}^lL_{lk}\mathbf{n} + \tfrac{\partial L_{ij}}{\partial u^k}\mathbf{n} \text{ (make tangent comp via } \mathbf{x}_l\text{)}\\
&= \left(\tfrac{\partial\Gamma_{ij}^l}{\partial u^k} + \Gamma_{ij}^p\Gamma_{pk}^l - L_{ij}L_{pk}g^{pl}\right)\mathbf{x}_l \ + \ \left(\Gamma_{ij}^lL_{lk} + \tfrac{\partial L_{ij}}{\partial u^k}\right)\mathbf{n} \text{ (factor } x_l \text{ and } \mathbf{n}\text{)}
\end{aligned}$$

Interchanging j and k we obtain

$$\mathbf{x}_{ikj} = \left(\frac{\partial\Gamma_{ik}^l}{\partial u^j} + \Gamma_{ik}^p\Gamma_{pj}^l - L_{ik}L_{pj}g^{pl}\right)\mathbf{x}_l \ + \ \left(\Gamma_{ik}^lL_{lj} + \frac{\partial L_{ik}}{\partial u^j}\right)\mathbf{n}$$

Since $\mathbf{x}_{ijk}=\mathbf{x}_{ikj}$, both the tangent and the normal components of $\mathbf{x}_{ijk}-\mathbf{x}_{ikj}$ are zero. The coefficient of the tangent component is

$$\frac{\partial\Gamma_{ij}^l}{\partial u^k} + \Gamma_{ij}^p\Gamma_{pk}^l - L_{ij}L_{pk}g^{pl} - \frac{\partial\Gamma_{ik}^l}{\partial u^j} - \Gamma_{ik}^p\Gamma_{pj}^l + L_{ik}L_{pj}g^{pl} = L_{ik}L_{pj}g^{pl} - L_{ij}L_{pk}g^{pl} - R_{ijk}^l = 0.$$

This proves the Gauss's equations.

The coefficient of the normal component is

$$\Gamma_{ij}^lL_{lk} + \frac{\partial L_{ij}}{\partial u^k} - \Gamma_{ik}^lL_{lj} - \frac{\partial L_{ik}}{\partial u^j} = 0$$

proving Codazzi-Mainardi equations. QED.

We can now present the proof of Theorema Egregium.

Theorema Egregium. The Gaussian K is dependent solely on the coefficient of the first fundamental form and their derivatives by

$$\boxed{K = \frac{g_{1i} R^i_{212}}{g}.}$$

Proof. Multiplying Gauss's equation $R^l_{ijk} = L_{ik}L_{jp}g^{pl} - L_{ij}L_{kp}g^{pl}$ by g_{lm}, we obtain $R^l_{ijk}g_{lm} = L_{ik}L_{jp}g^{pl}g_{lm} - L_{ij}L_{kp}g^{pl}g_{lm} = (L_{ik}L_{jp} - L_{ij}L_{kp})\delta^p_m = L_{ik}L_{jm} - L_{ij}L_{km}$. Taking $i = k = 2$, and $j = m = 1$, we obtain $L = L_{22}L_{11} - L_{21}L_{21} = R^l_{212}g_{l1}$.

From here we have that $K = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}L_{21}}{g} = \frac{R^l_{212}g_{l1}}{g}$. QED.

The curvature tensor R^l_{ijk} plays the key role when generalizing the results of this section to higher dimensions. This tensor can be viewed as a mapping of three vectors of the tangent space onto the tangent space itself given by

$$R(X,Y)Z \mapsto R^l_{ijk}X^jY^kZ^i\mathbf{x}_l$$

This formula relates to the curvature since $K = \frac{(R(\mathbf{x}_2,\mathbf{x}_1)\mathbf{x}_1)\cdot\mathbf{x}_2}{|\mathbf{x}_2\times\mathbf{x}_1|^2}$. The tensor R measures the curvature on the following way. Note that when a vector in space is parallel transported around a loop in a plane, it will always return to its original position. The Riemann curvature tensor directly measures the failure of this on a general surface. This failure is known as the holonomy of the surface.

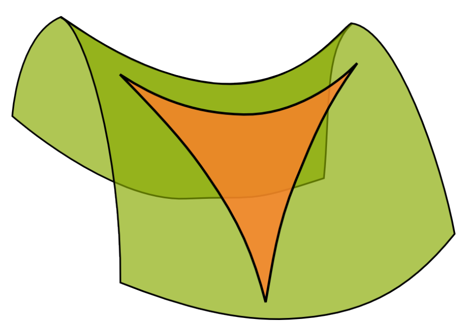
The connection between the areas and angles on a surface (thus the coefficients g_{ij}) and its Gaussian curvature can be seen in the following. The surface integral of the Gaussian curvature over some region of a surface is called the **total curvature**.

The total curvature of a geodesic triangle equals the deviation of the sum of its angles from 180 degrees. In particular,

- On a surface of total curvature zero, (such as a plane for example), the sum of the angles of a triangle is precisely 180 degrees.
- On a surface of positive curvature, the sum of angles of a triangle exceeds 180 degrees. For example, consider a triangle formed by the equator and two meridians on a sphere. Any meridian intersects the equator by 90 degrees. However, if the angle between the two meridians is $\theta > 0$, then the sum of the angles in the triangle is $180 + \theta$ degrees. In the figure on the right, the angles add to 270 degrees.
- On a surface of negative curvature, the sum of the angles of a triangle is less than 180 degrees.

$K = 0 \Rightarrow \sum$ angles $= 180$,

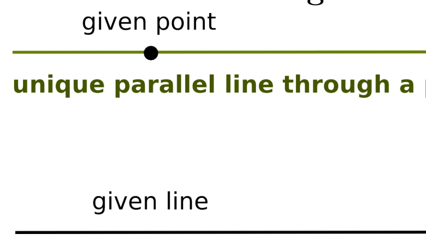
$K > 0 \Rightarrow \sum$ angles > 180,



$K < 0 \Rightarrow \sum$ angles < 180

The curvature impact also the number of lines passing a given point, parallel to a given line. Surfaces for which this number is not equal to one are models of **non-Euclidean geometries**.

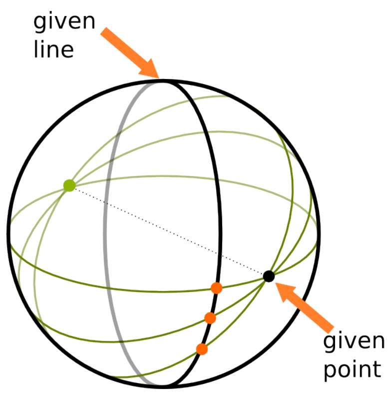
Recall that the parallel postulate in **Euclidean geometry** is stating that in a plane there is exactly one line passing a given point that does not intersect a given line, i.e. there is **exactly one line** parallel to a given line passing a given point.



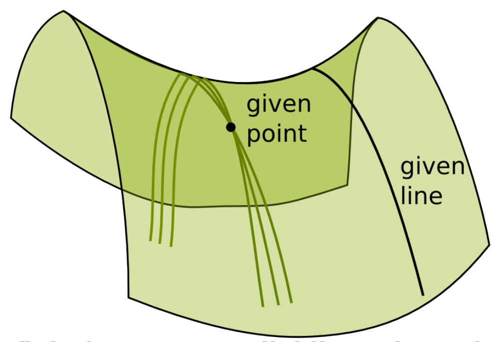
unique parallel line through a point

In **elliptic geometry** the parallel postulate is replaced by the statement that there is **no line** through a given point parallel to a given line. In other words, all lines intersect.

In **hyperbolic geometry** the parallel postulate is replaced by the statement that there are at least two distinct lines through a given point that do not intersect a given line. As a consequence, there are **infinitely many lines** parallel to a given line passing a given point.



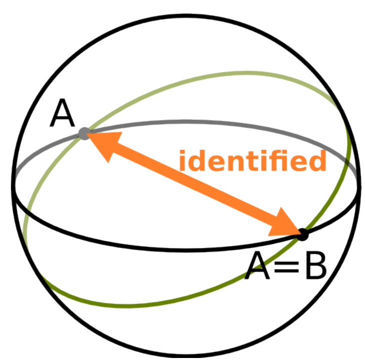
no parallel lines through a point



infinitely many parallel lines through a point

We present the projective plane which is a model of elliptic geometry and Poincaré half plane and disc which are models of hyperbolic geometry.

The projective plane $\mathbb{R}P^2$ is defined as the image of the map that identifies antipodal points of the sphere S^2. More generally, n-projective plane $\mathbb{R}P^n$ is defined as the image of the map that identifies antipodal points of the n-sphere S^n.

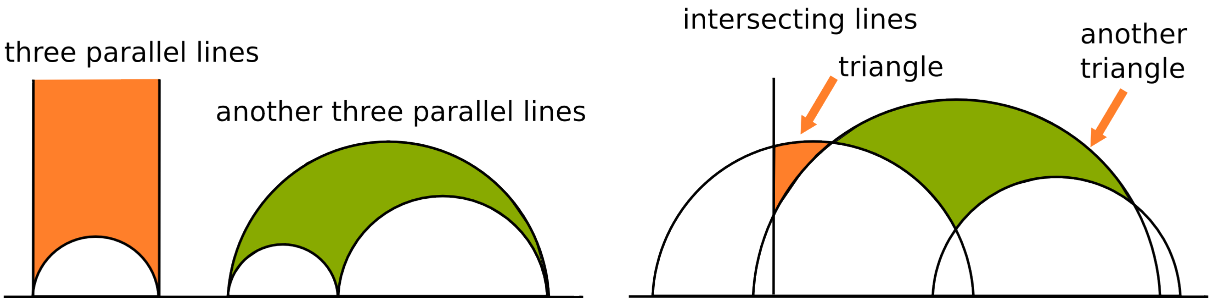


While there are lines which do not intersect (i.e. parallel lines) in a regular plane, every two "lines" (great circles on the sphere with antipodal points identified) in the projective plane intersect in one and only one point. This is because every pair of great circles intersect in exactly two points antipodal to each other. After the identification, the two antipodal points become a single point and hence every two "lines" of the projective plane intersect in a single point. The standard metric on the sphere gives rise to the metric on the projective plane. In this metric, the curvature K is positive.

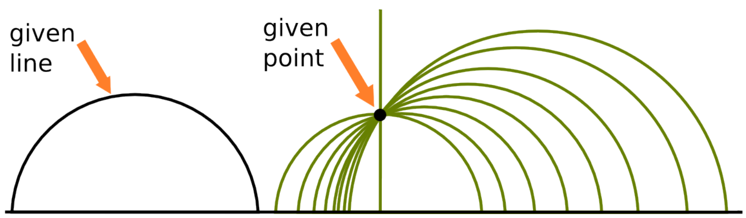
The projective plane can also be represented as the set of lines in $\mathbb{R}^3$ passing the origin. The distance between two such elements of the projective plane is the angle between the two lines in $\mathbb{R}^3$. The "lines" in the projective planes are the planes in $\mathbb{R}^3$ that pass the origin. Every two such "lines" intersect at a point (since every two planes in $\mathbb{R}^3$ that contain the origin intersect in a line passing the origin).


All lines intersect

Poincaré half-plane. Consider the upper half $y > 0$ of the plane $\mathbb{R}^2$ with metric given by the first fundamental form $\begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}$. In this metric, the geodesic (i.e. the "lines") are circles with centers on x-axis and half-lines that are perpendicular to x-axis.

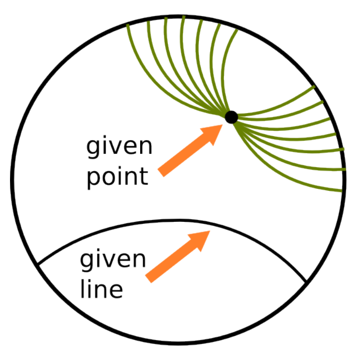


Given one such line and a point, there is more than one line passing the point that does not intersect the given line. In the given metric, the Gaussian curvature is negative.



infinitely many parallel lines through a point

Poincaré disc. Consider the disc $x^2 + y^2 < 1$ in $\mathbb{R}^2$ with metric given by the first fundamental form $\begin{bmatrix} \frac{1}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{1}{(1-x^2-y^2)^2} \end{bmatrix}$. In this metric, the geodesic (i.e. the "lines") are diameters of the disc and the circular arcs that intersect the boundary orthogonally. Given one such "line" and a point in the disc, there is more than one line passing the point that does not intersect the given line. In the given metric, K is negative.



infinitely many parallel lines through a point

Fundamental Theorem of Surfaces

The coefficients of curvature tensor R^l_{ijk} are defined via the Christoffel symbols Γ^k_{ij} and the Christoffel symbols can be computed via the first fundamental form using the formula

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}.$$

Thus, both the coefficients of the curvature tensor and the Christoffel symbols are completely determined by the first fundamental form. In light of this fact, Gauss's and Codazzi-Mainardi equations can be viewed as equations connecting the coefficients of the first fundamental form g_{ij} with the coefficients of the second fundamental form L_{ij}.

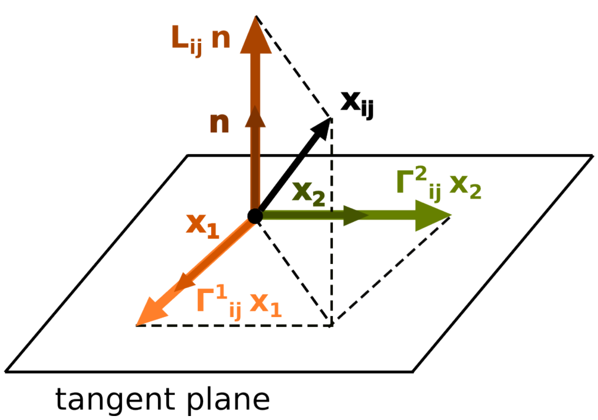
The Fundamental Theorem of Surfaces states that a surface is uniquely determined by **the coefficients of the first and the second fundamental form.** More specifically, if g_{ij} and L_{ij} are symmetric functions (i.e. $g_{ij} = g_{ji}$ and $L_{ij} = L_{ji}$) such that $g_{11} > 0$ and $g > 0$, and such that both Gauss's and Codazzi-Mainardi equations hold, there is a coordinate patch $\mathbf{x}$ such that g_{ij} and L_{ij} are coefficients of the first and the second fundamental form respectively. The patch $\mathbf{x}$ is unique up to a rigid motion (i.e. rotations and translations in space).

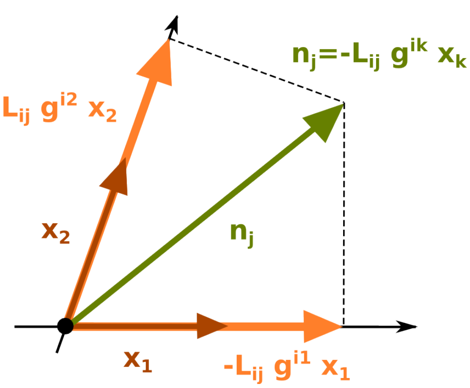
The idea of the proof of this theorem is similar to the proof of the Fundamental Theorem of Curves. Namely, note that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in tangent plane are independent by assumption that the patch is proper. Moreover, the vector $\mathbf{n}$ is independent of $\mathbf{x}_1$ and $\mathbf{x}_2$ since it is not in the tangent plane. Thus the three vectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{n}$ represent a "moving frame" of the surface analogous to the moving frame $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ of a curve.

Gauss formula and Weingarten's equations represent (partial) differential equations relating the derivatives of $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{n}$ in terms of the three vectors themselves

Gauss formula $\qquad \mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + L_{ij}\mathbf{n}.$

Weingarten's equations $\qquad \mathbf{n}_j = -L_{ij} g^{ik} \mathbf{x}_k$

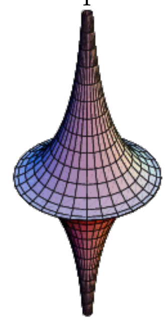




As opposed to a system of ordinary differential equations, there is no theorem that guarantees an existence and uniqueness of a solution of a system of partial differential equations. However, in case of the equations for $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{n}$, the existence and uniqueness of solution follows from the fact that both Gauss's and Codazzi-Mainardi equations hold. Thus, the apparatus $\mathbf{x}_1, \mathbf{x}_2$, $\mathbf{n}$, g_{ij}, L_{ij} describes a surface.

Examples. It can be shown that a surface of revolution obtained by revolving the unit speed curve $(r(s), z(s))$ about z-axis, has Gaussian curvature K equal to $\frac{-r''}{r}$. If K is constant, this yields a differential equation $r'' + Kr = 0$ that can be solved for r. Then z can be obtained from the condition that $z'^2 + r'^2 = 1$ i.e. from $z = \pm \int_0^s \sqrt{1 - r'^2} ds$. Thus, all the surfaces of revolution of constant curvature can be characterized on this way. We distinguish three cases:

- $K = a^2 > 0 \Rightarrow r'' + a^2 r = 0 \Rightarrow r(s) = c_1 \cos as + c_2 \sin as = A \cos as$. Sphere and outer part of the torus are in this group of surfaces.
- $K = 0 \Rightarrow r'' = 0 \Rightarrow r(s) = c_1 s + c_2$. Part of the plane, circular cylinder and circular cone are in this group.
- $K = -a^2 < 0 \Rightarrow r'' - a^2 r = 0 \Rightarrow r(s) = c_1 \cosh as + c_2 \sinh as$. Pseudo-sphere (see graph) and the inner part of the torus are in this group of surfaces.



Manifolds

A surface embedded in the 3-dimensional space $\mathbb{R}^3$ on a small enough scale resembles the 2-dimensional space $\mathbb{R}^2$. In particular, the inverse of a coordinate patch of a surface can be viewed as a mapping of a region on the surface to $\mathbb{R}^2$. This inverse is called an **atlas** or a **chart**.

Generalizing this idea to n-dimensions, we arrive to concept of a **n-dimensional manifold** or **n-manifolds** for short. Intuitively, an n-manifold locally looks like the space $\mathbb{R}^n$ – the neighborhood of every point on the manifold can be embedded in the space $\mathbb{R}^n$. The inverse of a coordinate patch of an n-manifold is a mapping of a region on the manifold to $\mathbb{R}^n$.

The coordinate patches are required to overlap smoothly on the intersection of their domains. The coordinate patches provide **local coordinates** on the n-manifold.

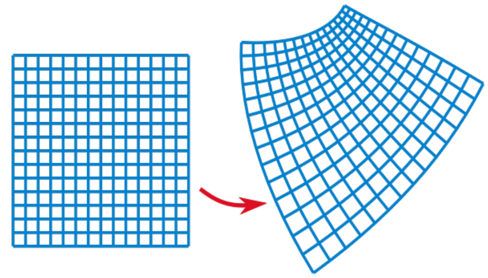
Making this informal definition rigorous, the concept of n-dimensional manifold is obtained. Considering manifolds instead of just surfaces has the following advantages:

1. Surfaces are 2-manifolds so this more general study of n-manifolds agrees with that of surfaces for $n = 2$.
2. A study of surfaces should not depend on a specific embedding in 3-dimensional space $\mathbb{R}^3$. The study of n-manifolds can be carried out without assuming the embedding into the space $\mathbb{R}^{n+1}$.
3. All the formulas involving indices ranging from 1 to 2 that we obtained for surfaces remain true for n-dimensional manifolds if we let the indices range from 1 to n.

When developing theory of manifolds one must be careful to develop all the concepts intrinsically and without any reference to extrinsic concepts. This underlines the relevance of Theorema Egregium in particular.

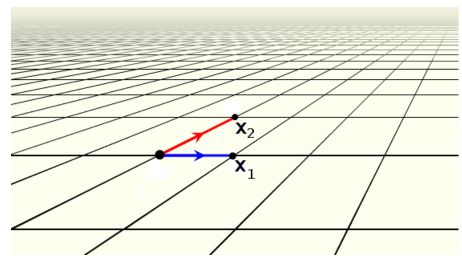
When considering manifolds for $n > 2$, one could make the following comment regarding the level of abstraction: is an n-manifold relevant for $n > 2$? Although the higher dimensional manifolds may not be embedded in three dimensional space, the theory of n-manifolds is used in high energy physics, quantum mechanics and relativity theory and, as such, is relevant. The Einstein space-time manifold, for example, has dimension four.

Coordinate Patches. Recall that a proper coordinate patch of a surface is given by parametric equations $\mathbf{x} = (x(u,v), y(u,v), z(u,v))$ such that x, y, z are one-to-one continuous functions with continuous inverses, continuous derivatives and such that $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$. It may not be possible to describe the whole surface with a



single coordinate patch but it is always be possible to cover the entire surface by "patching" several different coordinate patches together. Coordinate patches overlap smoothly provided that the following holds: if $\mathbf{x} = \mathbf{x}(u,v)$ is one coordinate patch defined on domain D and $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\bar{u}, \bar{v})$ is another defined on domain $\bar{D}$, then the composite functions $\mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^{-1} \circ \mathbf{x}$ are one-to-one and onto continuous functions on the intersection of D and $\bar{D}$ with continuous derivatives.

We define coordinate patches on n-manifold analogously: a coordinate patch of an non empty set of points M is a one-to-one mapping from an open region D in $\mathbb{R}^n$ into M given by $\mathbf{x}(u^1, u^2, \ldots, u^n)$. The n-tuple $(u^1, u^2, \ldots, u^n)$ represents the **local coordinates** on M. Two coordinate patches **overlap smoothly** provided that the following holds:



if $\mathbf{x}$ is one coordinate patch defined on domain D and $\bar{\mathbf{x}}$ is another defined on domain $\bar{D}$, then the composite functions $\mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^{-1} \circ \mathbf{x}$ are one-to-one and onto continuous functions on the intersection of D and $\bar{D}$ with continuous derivatives up to order at least three. For validity of the formulas involving the second fundamental form and equations from previous section, we may need the partial derivatives up to the third order to be continuous. If derivatives of any order are continuous, such function is said to be **smooth**.

We say that M is an n**-manifold** if there is a collection of coordinate patches such that:

1. The coordinate patches cover every point of M.
2. The coordinate patches overlap smoothly.
3. Every two different points on M can be covered by two different patches.
4. The collection of patches is maximal collection with conditions (1)–(3) (i.e. if a patch overlaps smoothly with every patch in collection is itself in collection).

Examples.

1. Euclidean space $\mathbb{R}^n$ is an n-manifold.
2. **Hypersurfaces**. Let f be a function with continuous derivatives that maps $\mathbb{R}^{n+1}$ into $\mathbb{R}$. The set of all vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ such that $f(\mathbf{x}) = 0$ defines an n-manifold usually referred to as hypersurface.

For example, the n**-plane** can be defined as the set of vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ such that

$$a_1x_1 + a_2x_2 + \ldots a_{n+1}x_{n+1} = b$$

for some constant vector $\mathbf{a} = (a_1, a_2, \ldots, a_{n+1})$. Note that the left side of the above equation is the dot product $\mathbf{a} \cdot \mathbf{x}$. Thus, this is a hypersurface with $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} - b$.

3. The n**-sphere** S^n is another example of a hypersurface. It can be defined as the set of vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}$ such that

$$x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$$

i.e. that $|\mathbf{x}|^2 = 1$. So, this is a hypersurface with $f(\mathbf{x}) = |\mathbf{x}|^2 - 1$.

4. n**-torus** T^n is defined as the set of vectors $\mathbf{x} = (x_1, \ldots, x_{2n})$ in $\mathbb{R}^{2n}$ such that $x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1, \ldots, x_{2n-1}^2 + x_{2n}^2 = 1$.

Partial Derivatives. Note that the differentiation is defined on the domain of a coordinate patch. When working with manifolds, we may want to be able to differentiate on the range of the coordinate patch as well. This can be done by considering derivative of a real-valued functions.

Let f be function that maps a neighborhood U of a point P on n-manifold M into a subset of $\mathbb{R}$. The function f is smooth if the composition $f \circ \mathbf{x}$ is smooth where $\mathbf{x}$ is a coordinate patch that contains P (thus meets U). Note that $f \circ \mathbf{x}$ is a function that maps domain D of $\mathbf{x}$ into $\mathbb{R}$. Thus, we can define the derivative of f with respect to coordinate u^i as $\frac{\partial f}{\partial u^i} = \frac{\partial (f \circ \mathbf{x})}{\partial u^i} \circ (\mathbf{x}^{-1})$ and use it to define the partial derivative operator at point P as

$$\frac{\partial}{\partial u^i}(P)(f) = \frac{\partial f}{\partial u^i}(P).$$

Directional Derivative and Tangent Vectors. Using the concept of partial derivative operator, we can define tangent vectors. In case of surfaces, tangent vectors were defined as velocity vectors of curves on surfaces. However, the definition of velocity vector is not available to us any more because it refers to embedding in $\mathbb{R}^3$. We can still define tangent vectors using an alternate route - via directional derivative. To understand the idea, consider a vector $\mathbf{v}$ in $\mathbb{R}^3$ given by (v^1, v^2, v^3). This defines a **directional derivative** operator by

$$D_{\mathbf{v}} = \mathbf{v} \cdot \nabla = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} + v^3 \frac{\partial}{\partial z}.$$

This operator is defined on the set of all real-valued functions f by $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f = v^1 \frac{\partial f}{\partial x} + v^2 \frac{\partial f}{\partial y} + v^3 \frac{\partial f}{\partial z}$. Thus, any linear combination of the partial derivatives can be considered as a directional derivative.

The set of all tangent vectors corresponds exactly to the set of all directional derivatives. For every curve $\boldsymbol{\gamma}(t)$ on the surface $\mathbf{x}(u^1, u^2)$, the velocity vector $\boldsymbol{\gamma}'(t)[f] = \frac{d}{dt}(f \circ \boldsymbol{\gamma})$ can be seen as an operator $\boldsymbol{\gamma}'(t)[f] = \frac{\partial (f \circ \mathbf{x})}{\partial u^i}(\mathbf{x}^{-1} \circ \boldsymbol{\gamma}(t)) \frac{du^i}{dt} = \frac{\partial f}{\partial u^i} \boldsymbol{\gamma}(t) \frac{du^i}{dt}$. Thus, $\boldsymbol{\gamma}(t)' = \frac{du^i}{dt} \frac{\partial}{\partial u^i}(\boldsymbol{\gamma}(t))$ is the directional derivative $D_{((u^1)', (u^2)')}$.

Having defined partial derivatives on an n-manifold M, allows us to define the tangent vectors at a point P as the set of all linear combinations of the partial derivatives $\frac{\partial}{\partial u^i}(P)$. Thus, $\mathbf{v}$ is a **tangent vector** if $\mathbf{v}$ is a linear combination of partial derivatives $\frac{\partial}{\partial u^i}(P)$ i.e. $\mathbf{v}$ is of the form $v^i \frac{\partial}{\partial u^i}(P)$.

The set of all tangent vectors is called the tangent space and is denoted by $T_P M$. This space represents the generalization of tangent plane of 2-manifolds. After showing that the partial derivatives are linearly independent, the vectors $\frac{\partial}{\partial u^i}(P)$ can be viewed as the basis of the tangent space $T_P M$.

Inner product and Metric. The coefficients of the first fundamental form are defined using the dot product of the basis vectors $\mathbf{x}_i = \frac{\partial}{\partial u^i}$. The familiar concept of the dot product generalizes to the **inner product**, a mapping of two vectors $\mathbf{v}$ and $\mathbf{w}$ producing a complex (or real) number $\mathbf{v} \cdot \mathbf{w}$ such that for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ and complex (or real) numbers a and b the following holds.

1. The product is symmetric: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

2. The product is bi-linear: $\mathbf{u} \cdot (a\,\mathbf{v} + b\,\mathbf{w}) = a\,\mathbf{u} \cdot \mathbf{v} + b\,\mathbf{u} \cdot \mathbf{w}$.

3. The product is positive definite: $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \vec{0}$.

For example, the dot product in $\mathbb{R}^n$ given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

If the tangent space $T_P M$ at every point P of an n-manifold M is equipped with an inner product, we say that M is a **Riemannian manifold** and the inner product is called a **Riemannian metric**. In this case, the coefficients of the first fundamental form at point P can be defined as

$$g_{ij}(P) = \frac{\partial}{\partial u^i}(P) \cdot \frac{\partial}{\partial u^j}(P).$$

With the inner product, the **length** of a vector $\mathbf{v}$ can be defined as $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. This enables defining the concepts of unit-length curves of a Riemannian n-manifold. Moreover, if g^{ij} denotes the matrix inverse to the matrix g_{ij}, then we can define **Christoffel Symbols** using the same formula which holds for surfaces

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}.$$

With the Christoffel symbols, we can also generalize the concept of **geodesic**. A curve $\boldsymbol{\gamma}$ on manifold M is geodesic if in each coordinate system defined along $\boldsymbol{\gamma}$ the equation $(u^k)'' + \Gamma^k_{ij}(u^i)'(u^j)' = 0$ holds for $k = 1, 2, \ldots n$. With this definition, a curve with the shortest possible length is necessarily a geodesic and every point P and every tangent vector $\mathbf{v}$ uniquely determine a geodesic $\boldsymbol{\gamma}$ with $\boldsymbol{\gamma}(0) = P$ and $\boldsymbol{\gamma}'(0) = \mathbf{v}$.

The Sectional Curvature. The Riemann curvature tensor can be defined via Christoffel symbols, using the same formula which holds for surfaces

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}$$

and the sectional curvature K at every point P of a manifold M can be defined as

$$K = \frac{g_{1i} R^i_{212}}{g}.$$

For 2-manifolds, the sectional curvature corresponds to Gaussian curvature. It is possible to define other types of curvatures: Ricci curvature and scalar curvature. These concepts are used in physics, especially in relativity theory.

There are many fascinating results in differential geometry. To conclude the exposition, we mention some of them:

1. If the coefficients of the Riemann curvature tensor are equal to zero, then the n-manifold is locally isometric to $\mathbb{R}^n$.

2. If a connected and complete Riemannian manifold of even dimension has constant sectional curvature $\frac{1}{a^2}$, then it is either a $2n$-sphere of radius a or a projective space.

3. A complete, connected and simply connected (every closed curve can be collapsed to a point) Riemannian manifold of constant sectional curvature c is

   - the sphere S^n of radius $\frac{1}{\sqrt{c}}$ if $c > 0$,
   - the space $\mathbb{R}^n$ if $c = 0$, and
   - a hyperbolic space if $c < 0$. The hyperbolic space H^n is the set of vectors in $\mathbb{R}^n$ of length smaller than 1 with the metric coefficients g_{ij} given at point $\mathbf{v}$ by $g_{ij}(\mathbf{v}) = \frac{4\delta_{ij}}{-c(1-|\mathbf{v}|^2)^2}$.

4. Poincaré conjecture (recently proven, see Wikipedia for more details): Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.