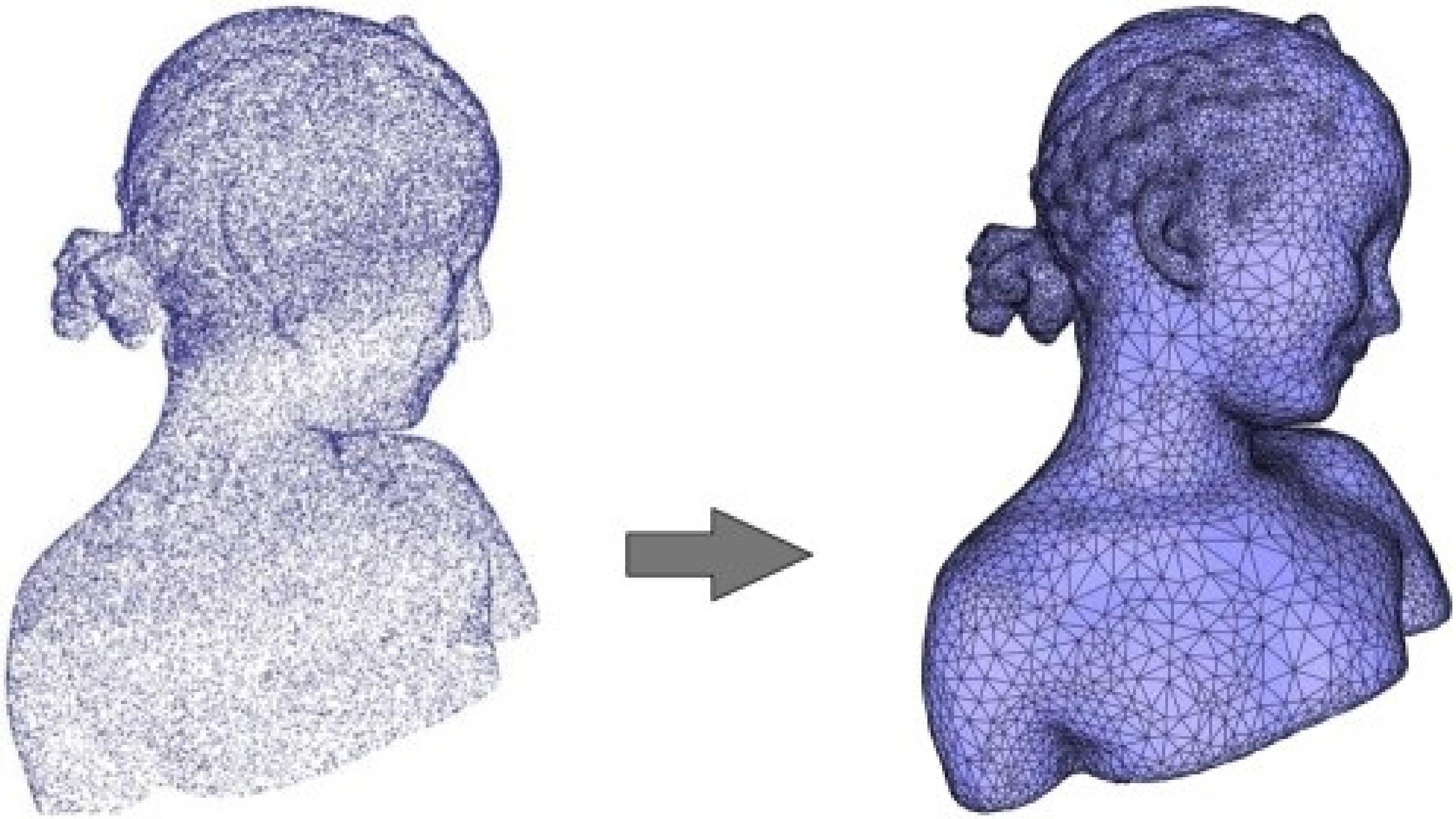
CGAL

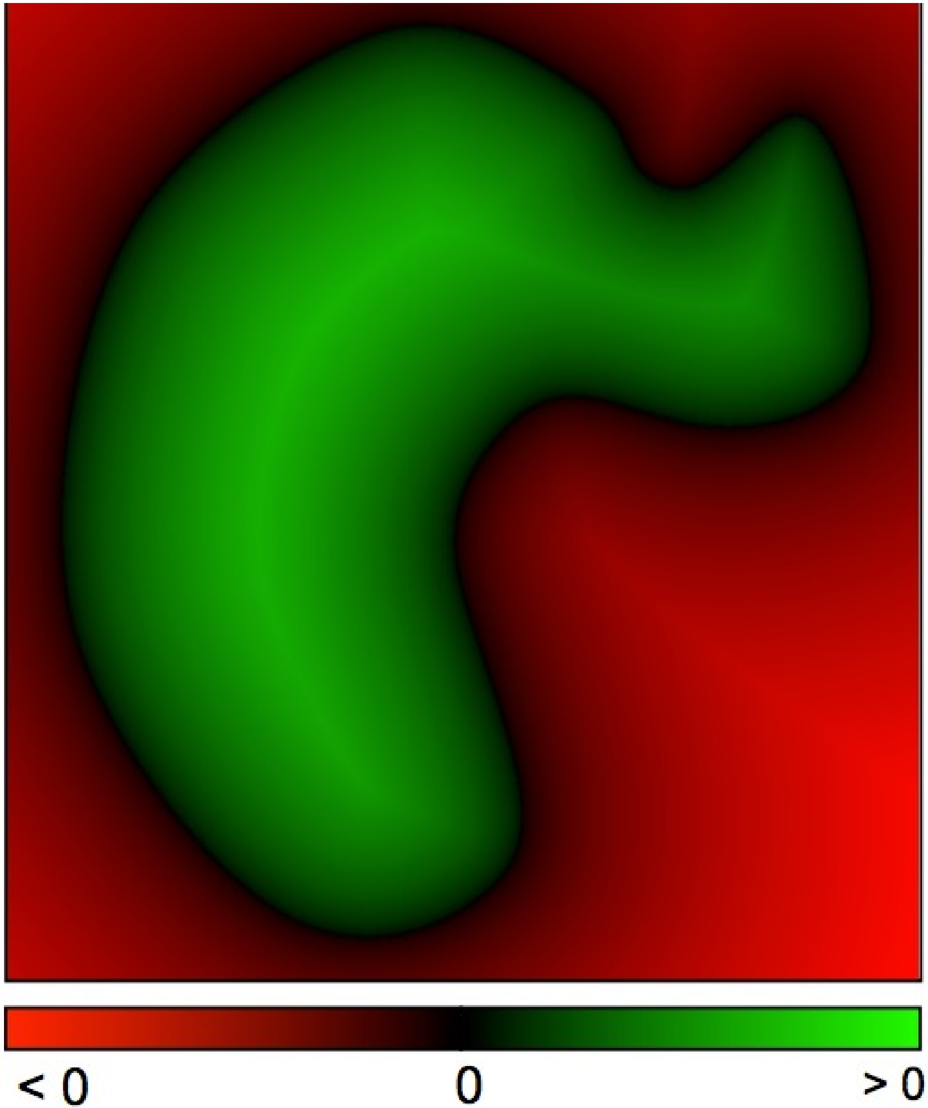


Poisson Surface Reconstruction

Siddhartha Chaudhuri http://www.cse.iitb.ac.in/~cs749

Recap: Implicit Function Approach

- Define a function with positive values inside the model and negative values outside

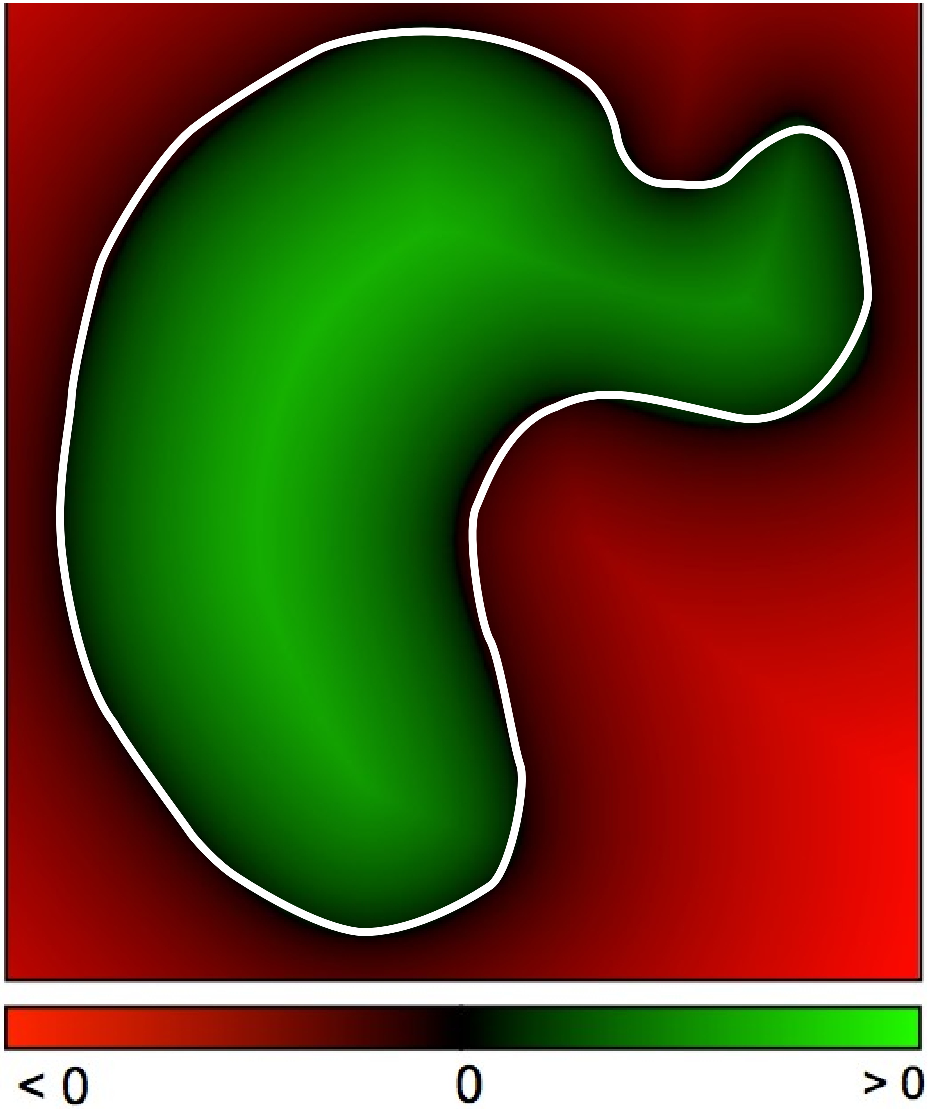


Slides adapted from Kazhdan, Bolitho and Hoppe

Recap: Implicit Function Approach

- Define a function with positive values inside the model and negative values outside

- Extract the zero-set



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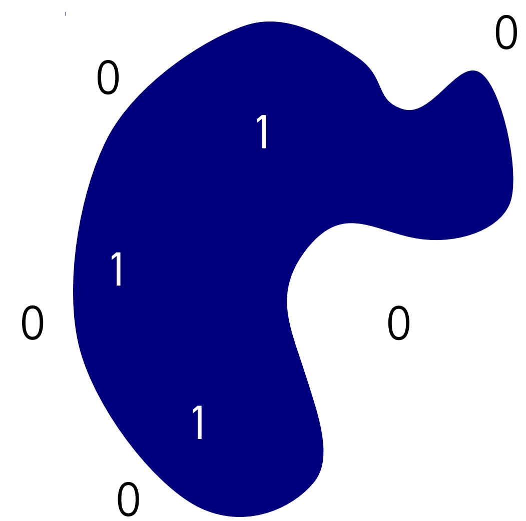
Recap: Key Idea

- Reconstruct the surface of the model by solving for the indicator function of the shape

$$\chi_M(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{if } p \notin M \end{cases}$$

In practice, we define the indicator function to be -1/2 outside the shape and 1/2 inside, so that the surface is the zero level set. We also smooth the function a little, so that the zero set is well defined.



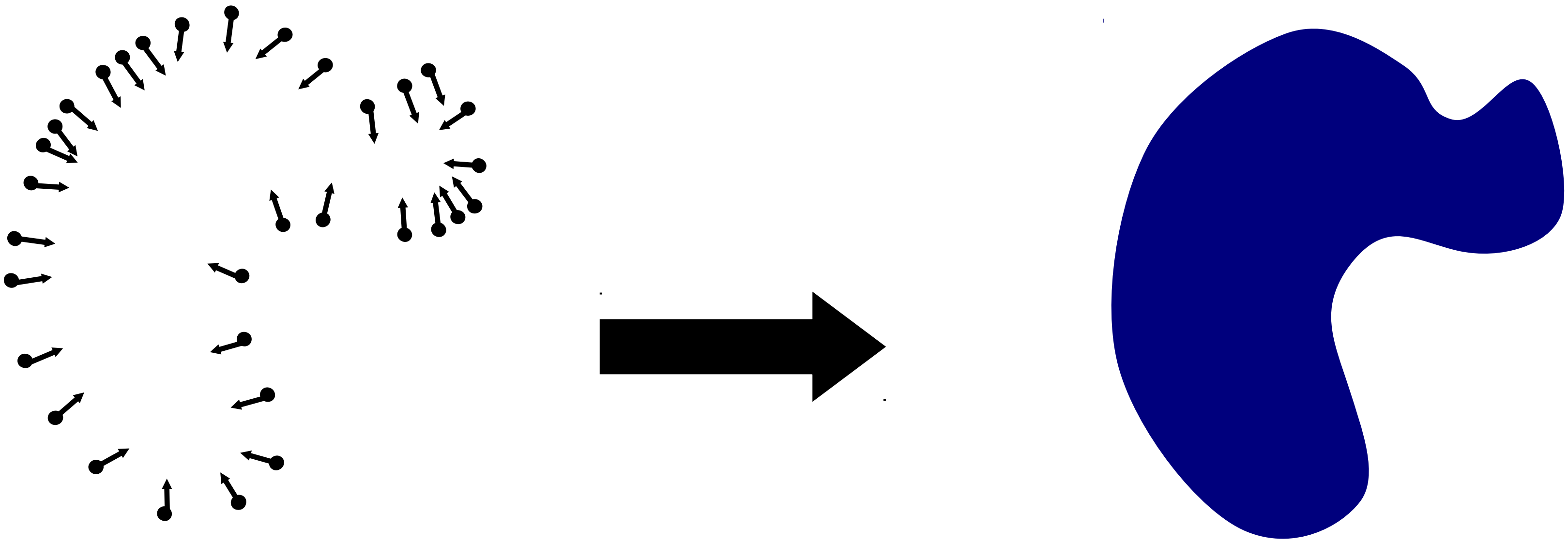
Indicator function

χ_M

Slides adapted from Kazhdan, Bolitho and Hoppe

Recap: Challenge

- How to construct the indicator function?



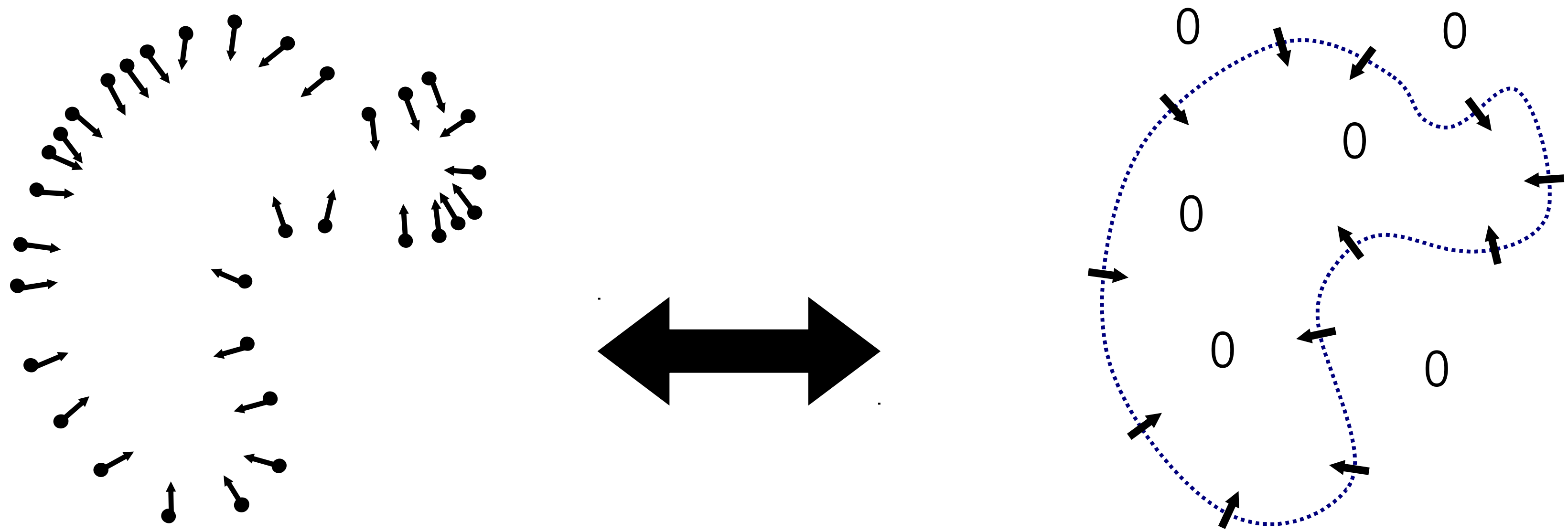
Oriented points

Indicator function
χ_M

Slides adapted from Kazhdan, Bolitho and Hoppe

Recap: Gradient Relationship

- There is a relationship between the normal field at the shape boundary, and the gradient of the (smoothed) indicator function



Oriented points

Indicator gradient

$\nabla \chi_M$

Slides adapted from Kazhdan, Bolitho and Hoppe

Operators

- Let's look at a 1D function $f : \mathbb{R} \rightarrow \mathbb{R}$
  - It has a first derivative given by

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$$

  - ... a second derivative, and a third...

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\frac{d}{dx} f \qquad \frac{d^3 f}{dx^3} = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx} f$$

- $\frac{d}{dx}$ is a general operation mapping functions to functions: it's called an **operator**
  - In fact, it's a **linear operator**: $\frac{d}{dx}(f+g) = \frac{d}{dx} f + \frac{d}{dx} g$

Variational Calculus

- Imagine we didn't know f, but we did know its derivative $g = \frac{df}{dx}$
- Solving for f is, obviously, integration

$$f = \int \frac{df}{dx} dx = \int g \, dx$$

- But what if g is not analytically integrable?
  - Then we can look for approximate solutions, drawn from some parametrized family of candidate functions

Variational Calculus

- Assume we have a family of functions F
- Let's minimize the mean squared approximation error over some interval Ω and functions $f \in F$

$$\text{minimize} \int_{\Omega} \left| \frac{df}{dx} - g \right|^2 dx$$

Euler-Lagrange Formulation

- **Euler-Lagrange equation**: Stationary points (minima, maxima etc) of a functional of the form

$$\int_{\Omega} L(x, f(x), f'(x))\, dx$$

$$f'(x) = \frac{df}{dx}$$

are obtained as solutions f to the PDE

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$$

Euler-Lagrange Formulation

- **Euler-Lagrange equation**: $\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$
- In our case, $L = (f'(x) - g(x))^2$, so

$$\frac{\partial L}{\partial f} = 0 \qquad \frac{\partial L}{\partial f'} = 2\left(f'(x) - g(x)\right)$$

$$\frac{d}{dx}\frac{\partial L}{\partial f'} = 2\left(f''(x) - g'(x)\right)$$

- Substituting, we get (a case of) the 1D **Poisson equation**:

$$f'' = g' \quad \text{or} \quad \frac{d^2 f}{dx^2} = \frac{dg}{dx}$$

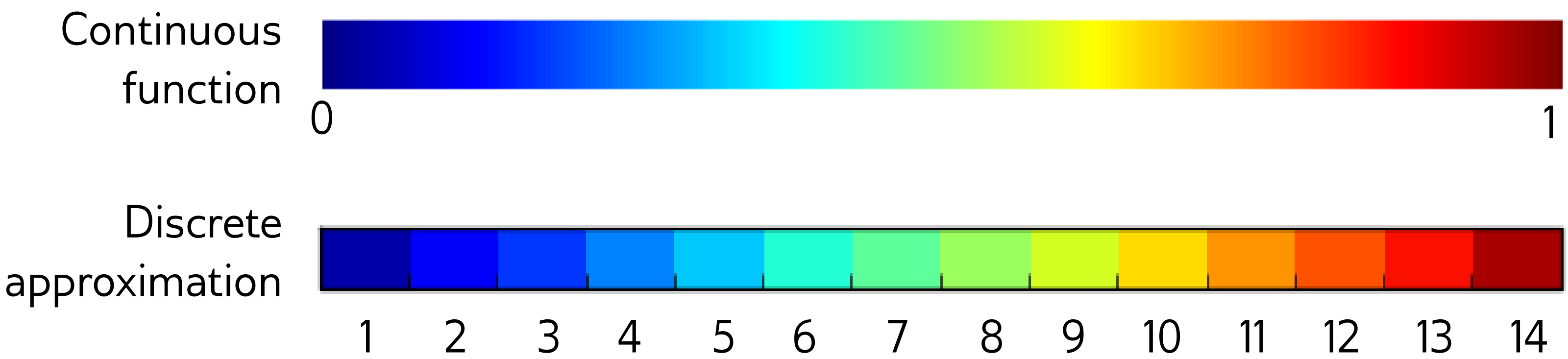
Link to Linear Least Squares

- Here, we want to minimize $\int_\Omega \left(f'(x) - g(x)\right)^2 dx$ and end up having to solve

$$\frac{d}{dx}\frac{d}{dx} f = \frac{d}{dx} g$$

  i.e. the two sides are equal at all points x
- Let's try to discretize this!
  - Sample n consecutive points $\{x_i\}$ from Ω
    - Assume (for simplicity) they're evenly spaced, so $x_{i+1} - x_i = h$
  - We want to minimize $\Sigma_i \, (f'(x_i) - g(x_i))^2$

Link to Linear Least Squares



- The derivative at x_i can be approximated as

$$f'(x_i) \approx \frac{f_{i+1} - f_i}{h} = \frac{1}{h}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix}$$

where f_i is shorthand for $f(x_i)$

Link to Linear Least Squares

- … and all the derivatives can be listed in one big matrix multiplication: $A\,\mathbf{f} = \mathbf{g}$, where

$$A = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ & 0 & -1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & -1 & 1 \\ 0 & 0 & \cdots & & 0 & -1 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix}$$

- $\mathbf{f}$ and $\mathbf{g}$ are discrete approximations of continuous functions f and g, and A is a discrete approximation for the continuous derivative operator $\frac{d}{dx}$!

Flashback

- Need to solve set of equations $A\mathbf{f} = \mathbf{g}$ in a least squares sense

$$\text{minimize } \|\mathbf{r}\|^2 = \|\mathbf{g} - A\mathbf{f}\|^2$$

- The directional derivative in direction $\delta\mathbf{f}$ is

$$\nabla\|\mathbf{r}\|^2 \,.\, \delta\mathbf{f} = 2\delta\mathbf{f}^{\mathrm{T}}(A^{\mathrm{T}}\mathbf{g} - A^{\mathrm{T}}A\mathbf{f})$$

- The minimum is achieved when all directional derivatives are zero, giving the normal equations

$$A^{\mathrm{T}}A\mathbf{f} = A^{\mathrm{T}}\mathbf{g}$$

- **Thought for the (Previous) Day:** Compare this equation to the Poisson equation

Link to Linear Least Squares

- Linear Least Squares: The $\mathbf{f}$ that minimizes $\|A\mathbf{f} - \mathbf{g}\|^2$ is the solution of $A^{\mathrm{T}}A\mathbf{f} = A^{\mathrm{T}}\mathbf{g}$
- Euler-Lagrange: The f that minimizes $\int_{\Omega}\left(\frac{df}{dx}(x)-g(x)\right)^2 dx$ is a solution of $\frac{d}{dx}\frac{d}{dx}f = \frac{d}{dx}g$
- Knowing that A is the discrete version of $\frac{d}{dx}$, everything lines up *except* for the transpose bit
  - How do we reconcile this?

Link to Linear Least Squares

- The derivative at x_i can *also* be approximated as

$$f'(x_i) \approx \frac{f_i - f_{i-1}}{h} = \frac{1}{h}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} f_{i-1} \\ f_i \end{bmatrix}$$

  … and derivatives at all x_i as $B\,\mathbf{f}$, where

$$B = \frac{1}{h}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ & -1 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & 1 & 0 \\ 0 & 0 & \cdots & & -1 & 1 \end{bmatrix}$$

  … which is just $-A^{\mathrm{T}}$!

- Can rewrite normal equations as $(-A^{\mathrm{T}})A\mathbf{f} = (-A^{\mathrm{T}})\mathbf{g}$

Uniqueness of Solutions

- The discrete operator A we constructed is full-rank (invertible), and gives a unique solution $A^{-1}\mathbf{g}$ for $\mathbf{f}$
- But the corresponding continuous problem has multiple solutions (e.g. if f is a solution, $(f + constant)$ is also a solution)
- Explanation: $A\mathbf{f} = \mathbf{g}$ implicitly imposes the boundary condition $f_n = -g_n$ (see the last row of the matrix)
  - In higher dimensions, the operator matrix A is non-square (maps scalar field to vector field) and not invertible. The system is overdetermined and we seek least-squares solutions

Discrete Second Derivative

- Multiplying the matrices, we get the discrete second derivative operator (the 1D Laplacian)

If you actually do the multiplication, this term is -1 and not -2. This is because our discretization does not correctly model the derivative at the end of the range. If you swap the matrices, the discrepancy occurs in the *last* element of the product instead.

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} \quad \text{discretized to} \quad \left(-A^T\right)A = \frac{1}{h^2}\begin{bmatrix} \boxed{-2} & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ & 1 & -2 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & -2 & 1 \\ 0 & 0 & \cdots & & 1 & -2 \end{bmatrix}$$

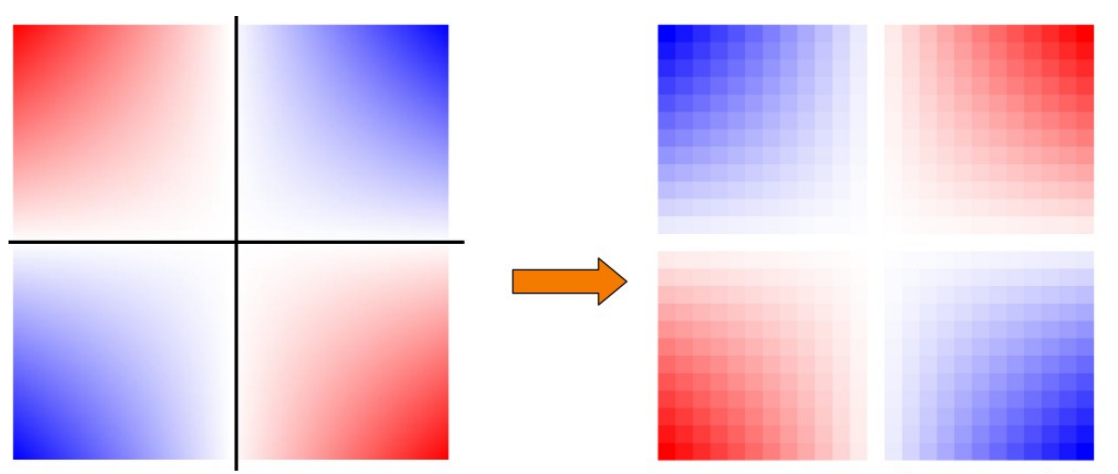
... which is the same as the Taylor series approximation for the second derivative

In higher dimensions

- We have a function $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$
- Differential operators (in 3D):
  - **Gradient** (of scalar-valued function): $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$
  - **Divergence** (of vector-valued function): $\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$
  - **Laplacian** (of scalar-valued function): $\Delta f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

In higher dimensions

- We have a function $f: \mathbb{R}^p \rightarrow \mathbb{R}^q$
  - We can discretize the domain as before, and obtain discrete analogues of the gradient ∇ (A), divergence $\nabla\cdot$ ($-A^T$) and Laplacian $\Delta = (\nabla\cdot)\nabla$ ($-A^T A$)
  - Note that the gradient and divergence matrices are no longer square (more on this next class)



$F(x,y) = xy$

$$F[i][j] = \frac{(i - w/2)\cdot(y - h/2)}{(w/2)\cdot(h/2)}$$

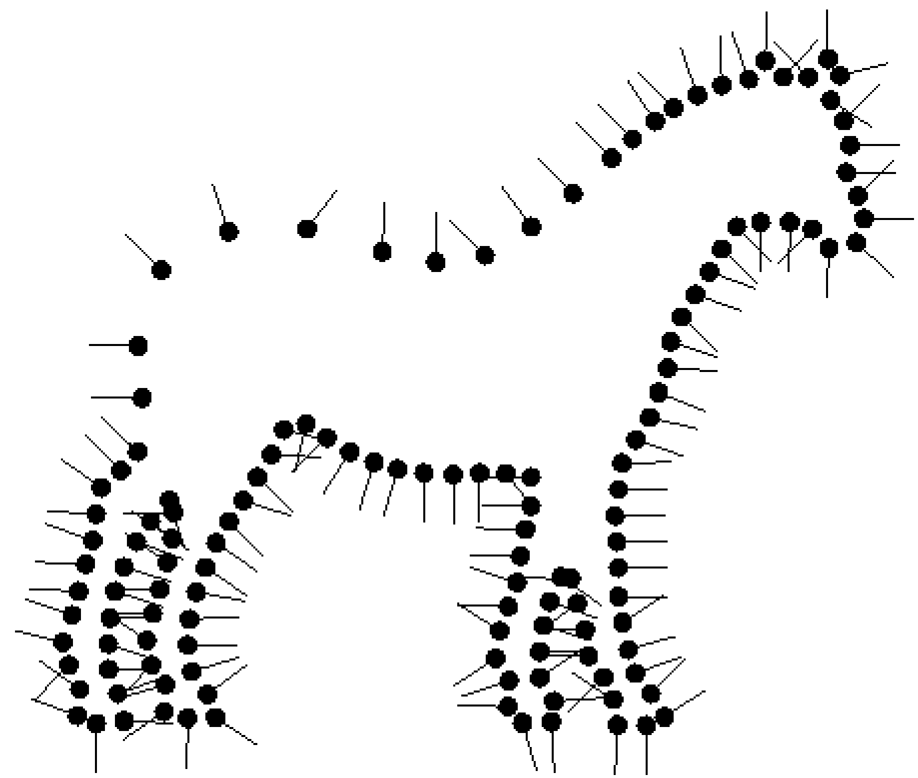
Takeaway

- A continuous variational problem can be approximated by a discrete one
  - Continuous function → Discrete vector of values
  - Continuous operator → Discrete matrix
  - Function composition → Matrix multiplication
  - Euler-Lagrange solution → Linear Least Squares
- Rest of this class: Overview of the pipeline of Poisson surface reconstruction
- Next class: The Galerkin approximation for recovering a continuous function from the discrete setup

Implementation

Given the Points:

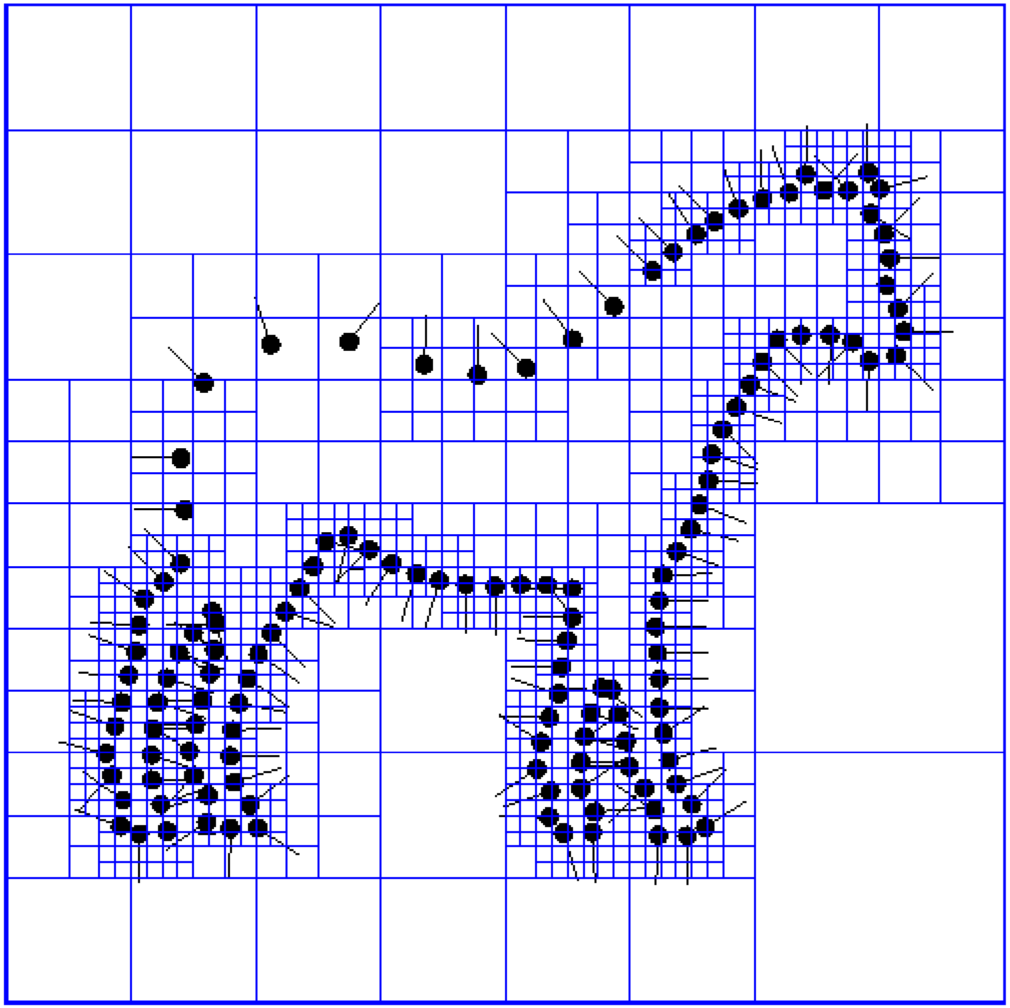
- Set octree
- Compute vector field
- Compute indicator function
- Extract iso-surface



Implementation: Adaptive Octree

Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
- Extract iso-surface

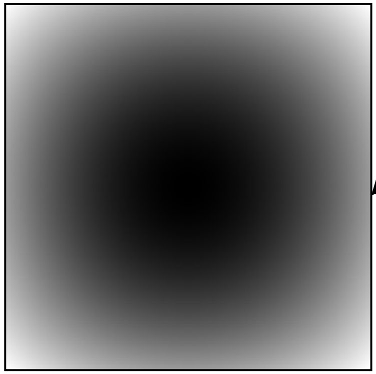


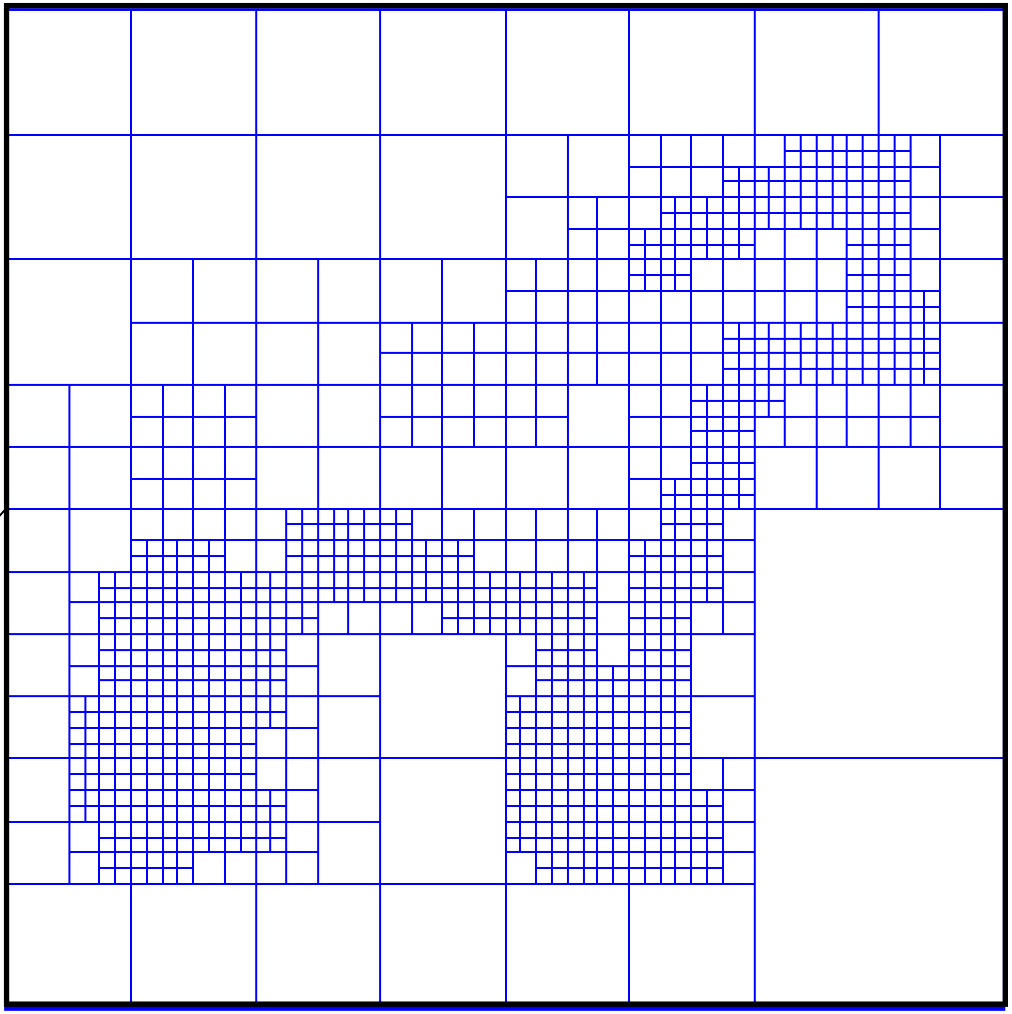
Slides adapted from Kazhdan, Bolitho and Hoppe

Implementation: Vector Field

Given the Points:

- Set octree
- Compute vector field
  - Define a function space
  - Splat the samples
- Compute indicator function
- Extract iso-surface





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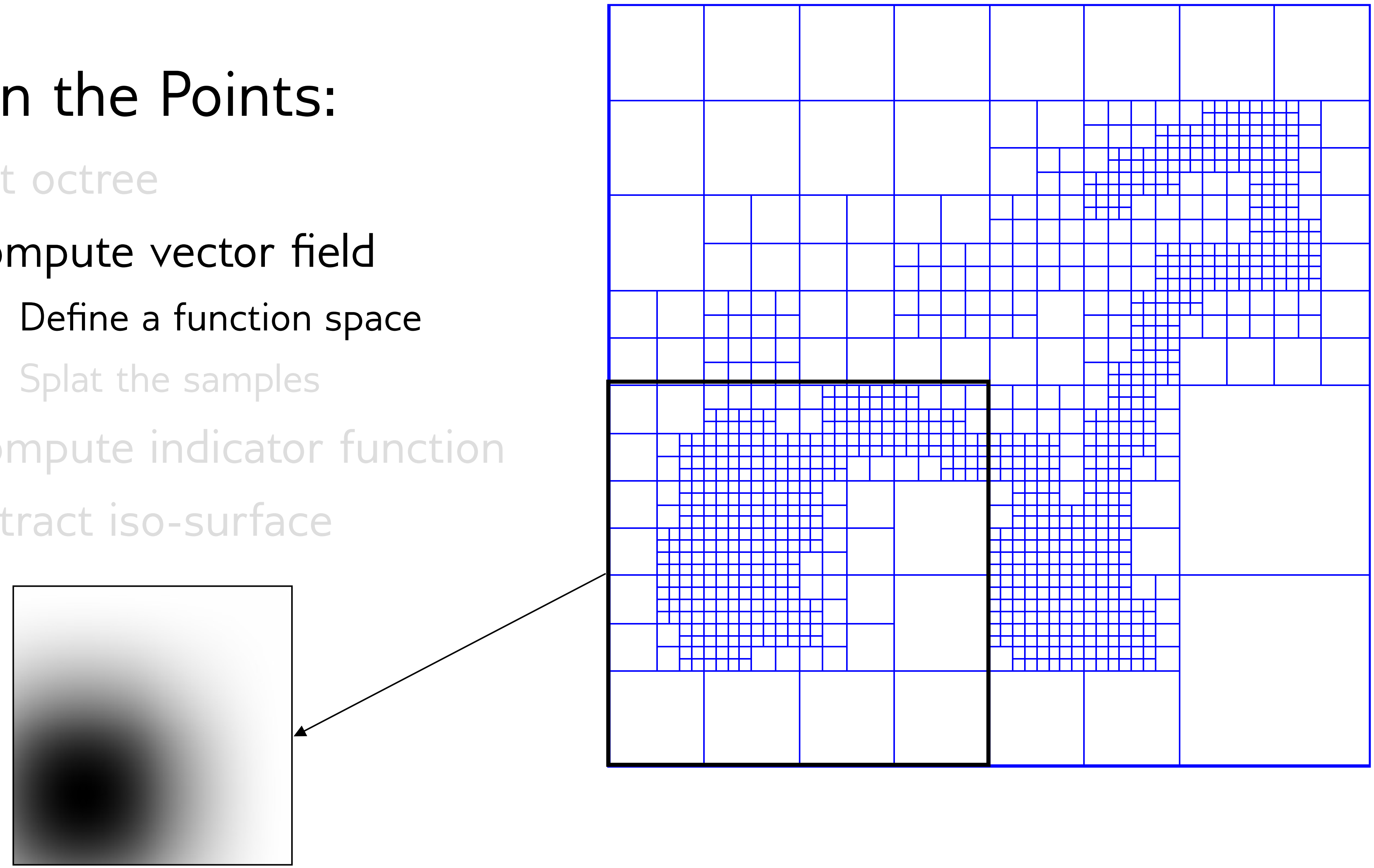

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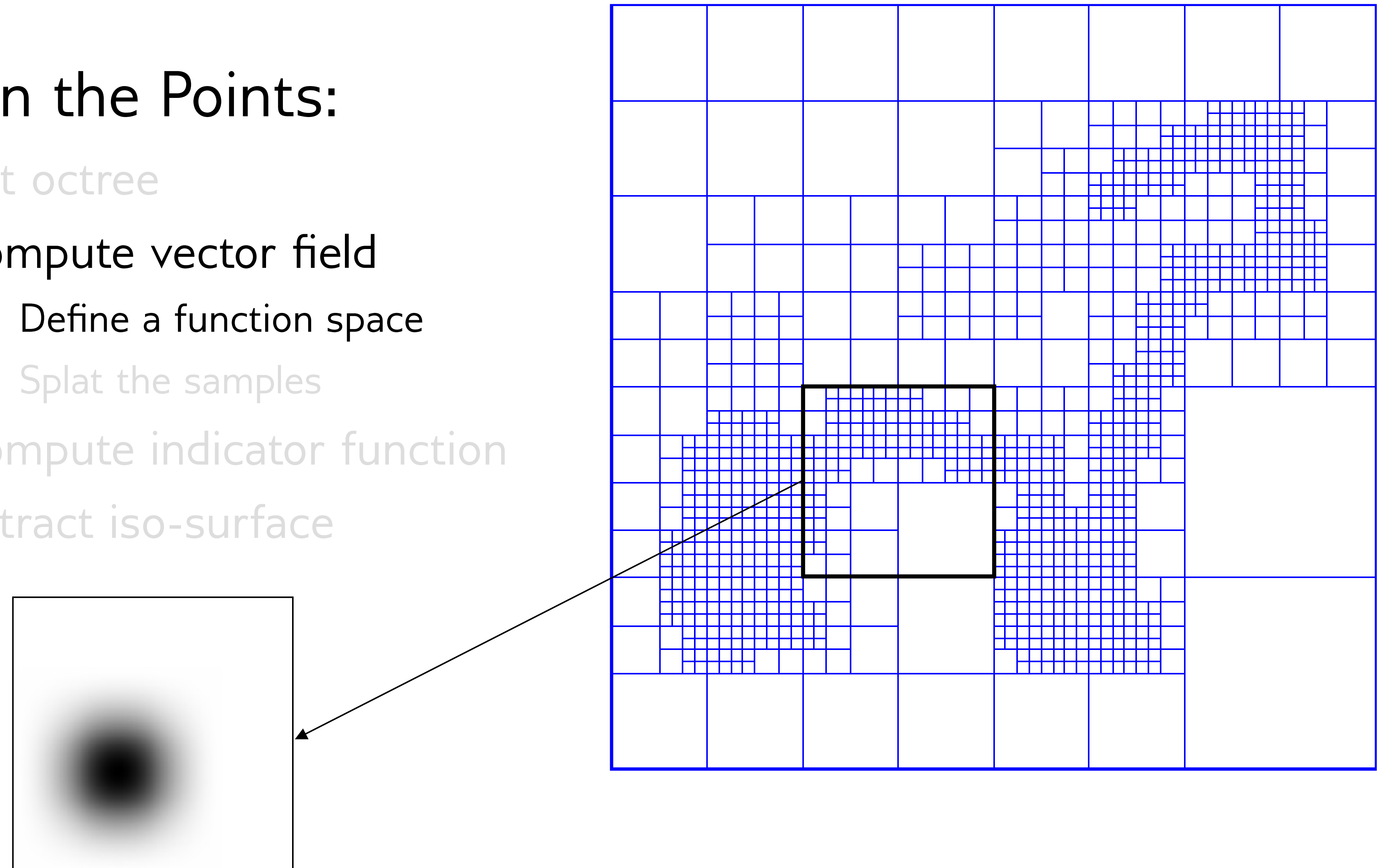

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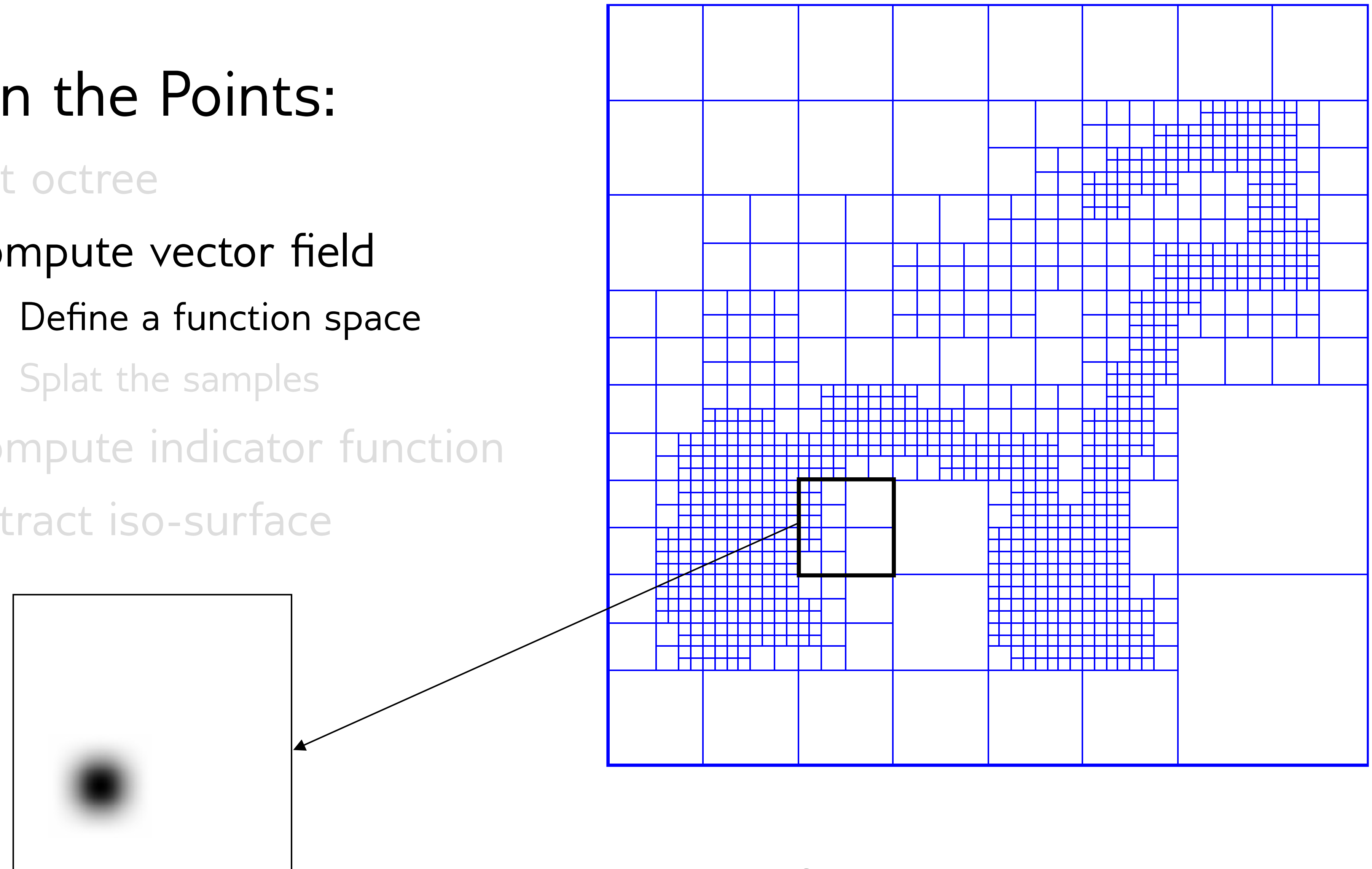

Implementation: Vector Field

Given the Points:

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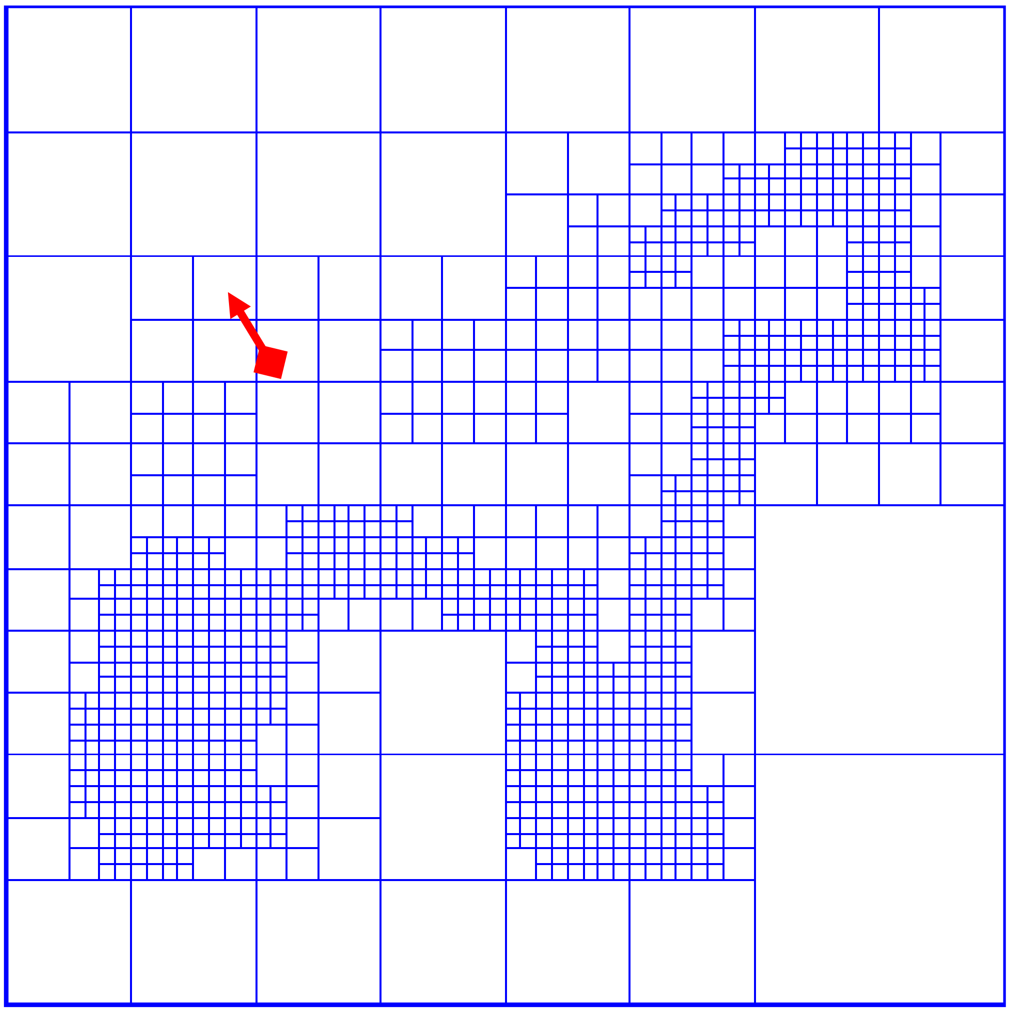
- Compute vector field
  - Define a function basis
  - Splat the samples
- Compute indicator function
- Extract iso-surface



Implementation: Vector Field

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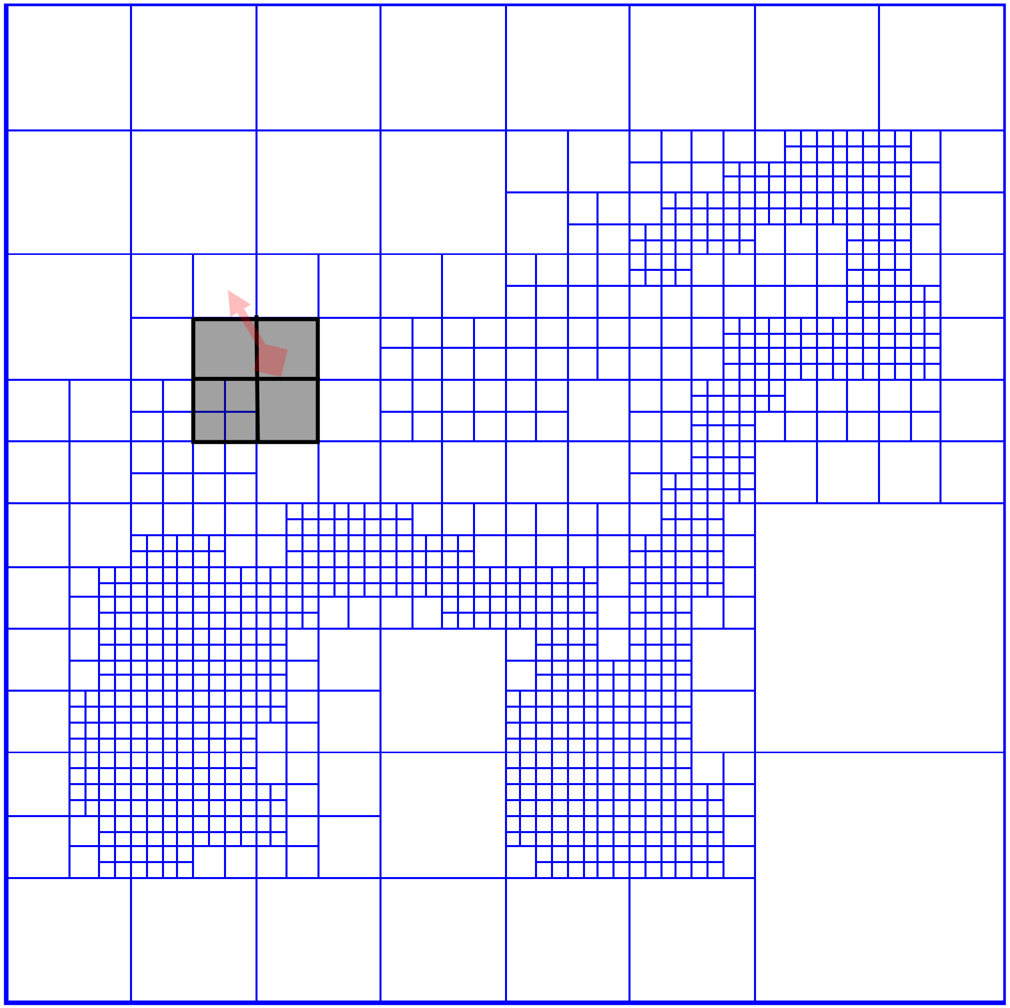

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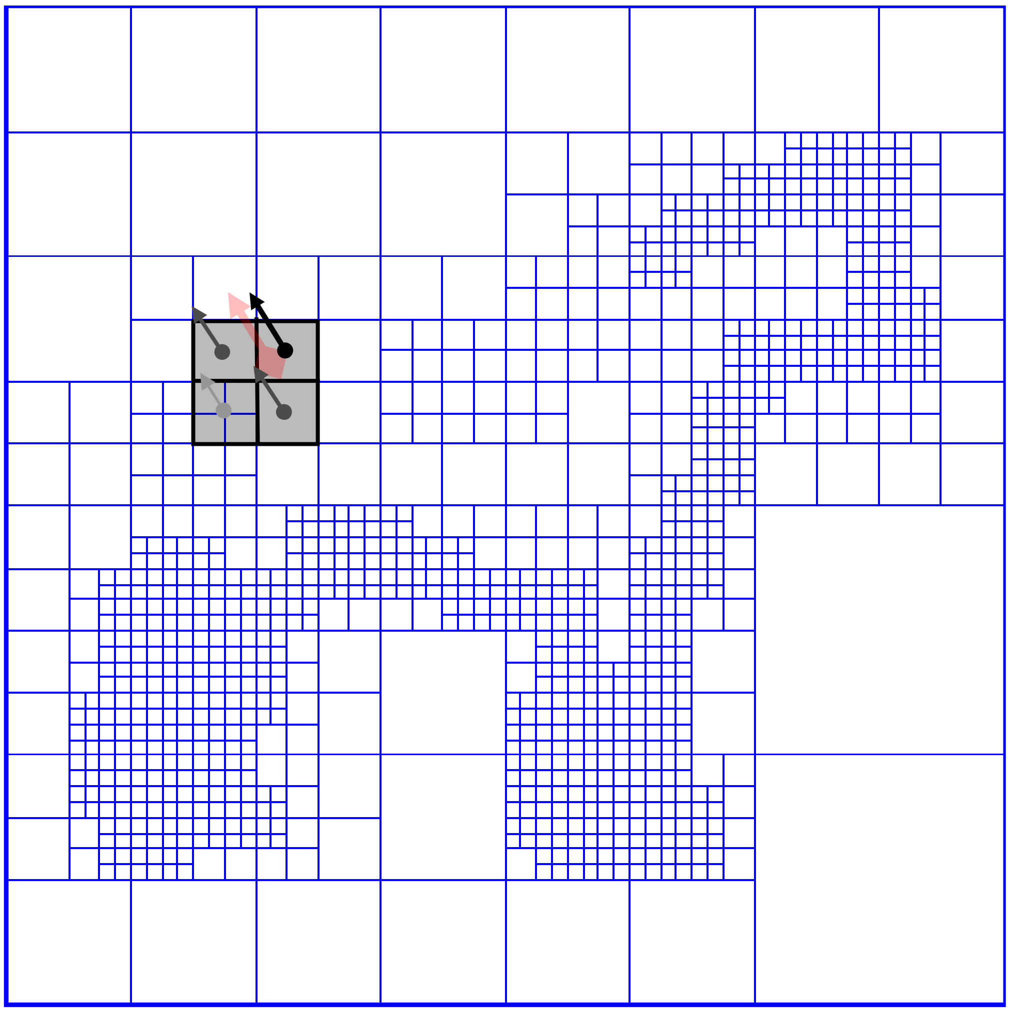

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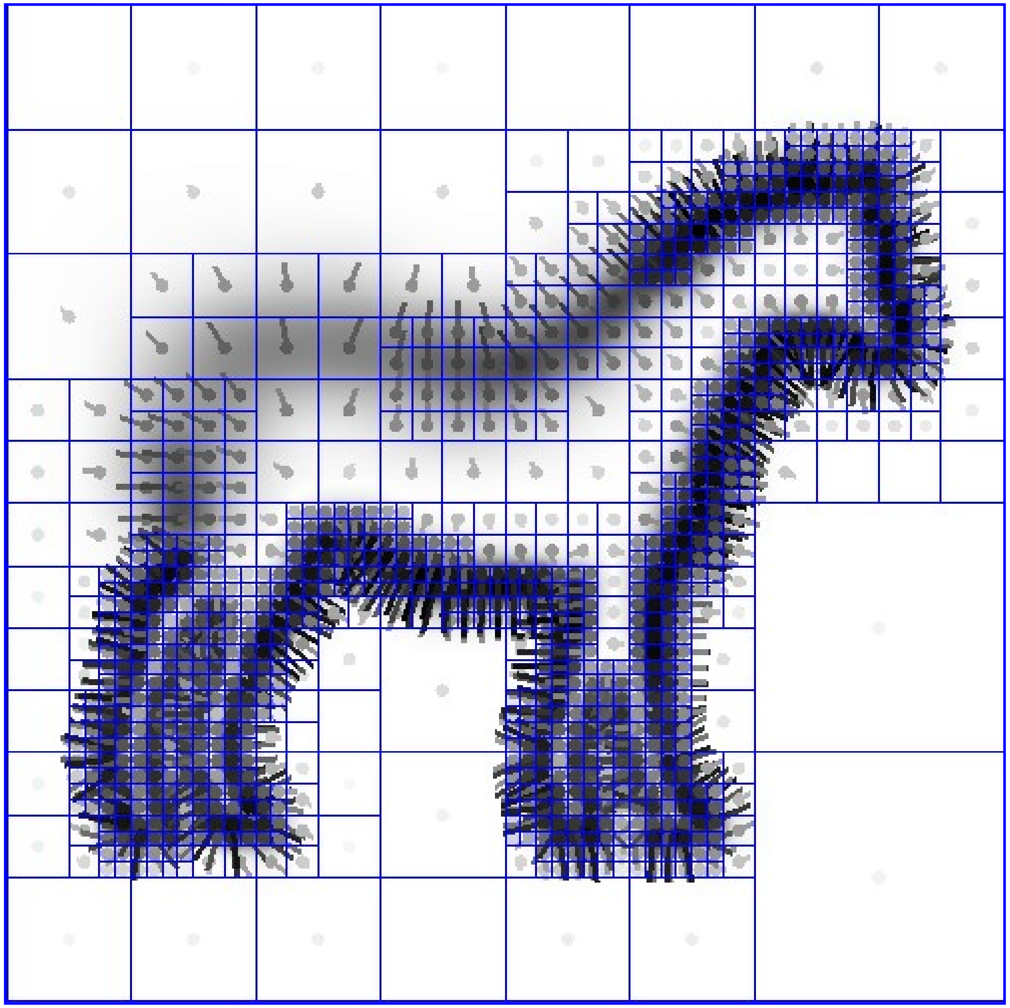


Slides adapted from Kazhdan, Bolitho and Hoppe

Implementation: Indicator Function

Given the Points:

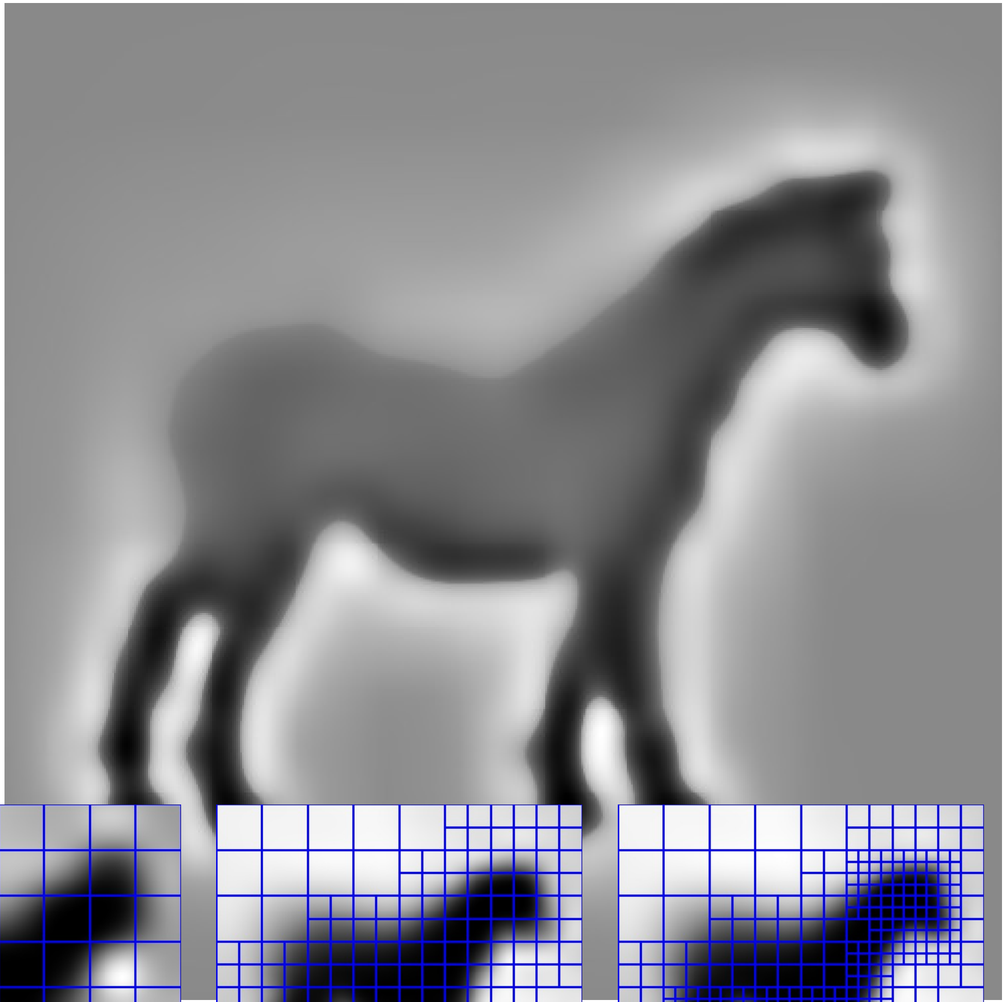
- Set octree
- Compute vector field
- Compute indicator function
  - Compute divergence
  - Solve Poisson equation
- Extract iso-surface

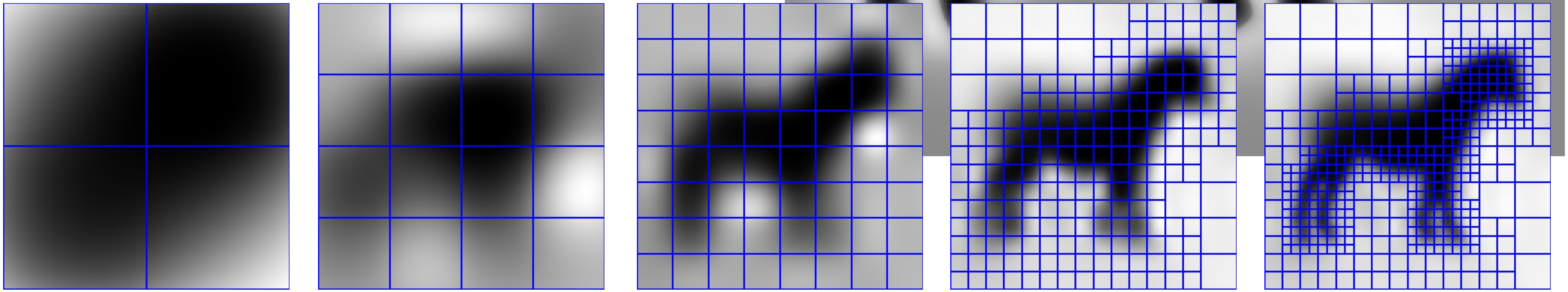
Slides adapted from Kazhdan, Bolitho and Hoppe

Implementation: Indicator Function

Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
  - Compute divergence
  - Solve Poisson equation
- Extract iso-surface





Implementation: Indicator Function

Given the Points:

- Set octree
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  - Solve Poisson equation
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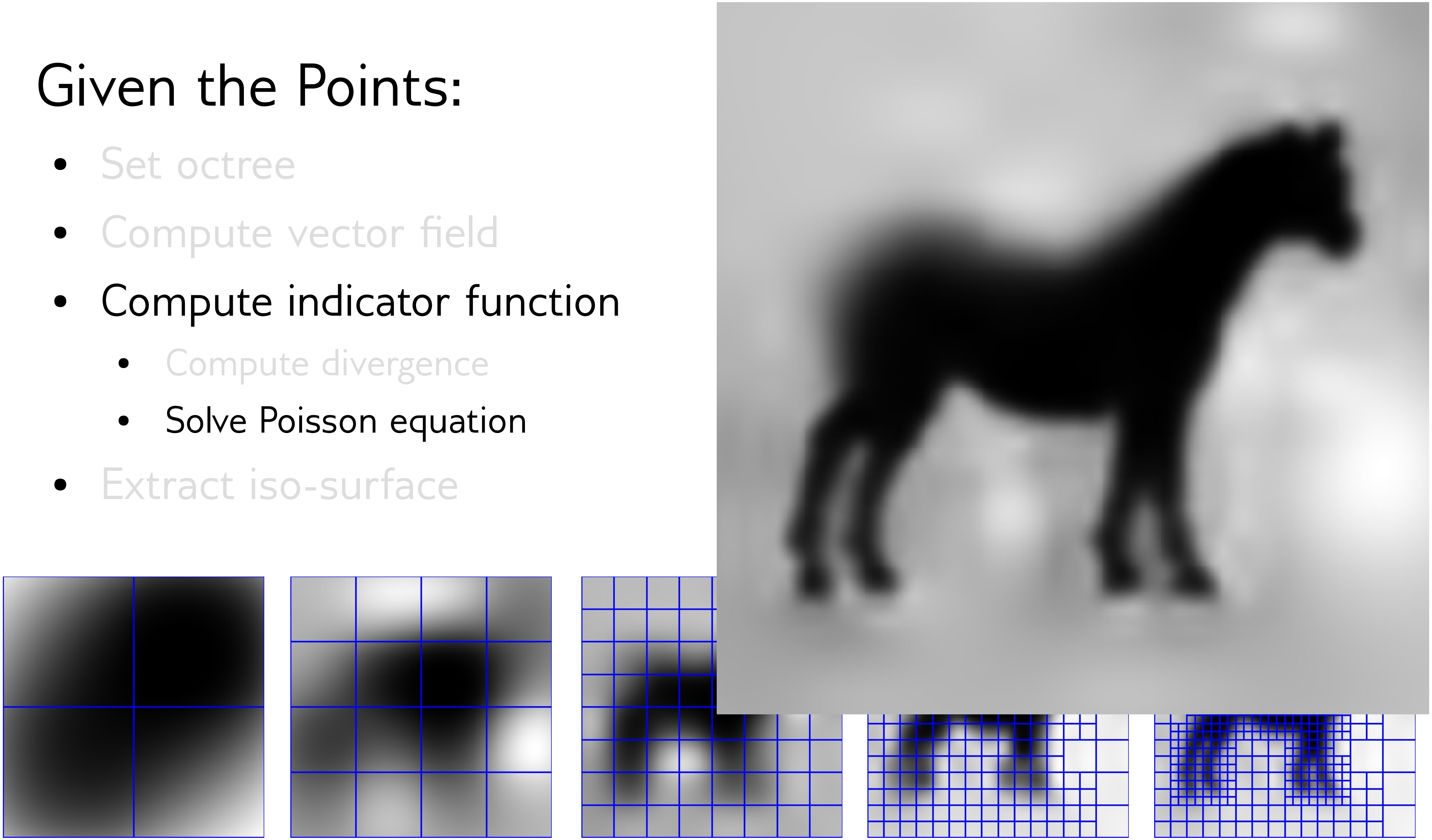
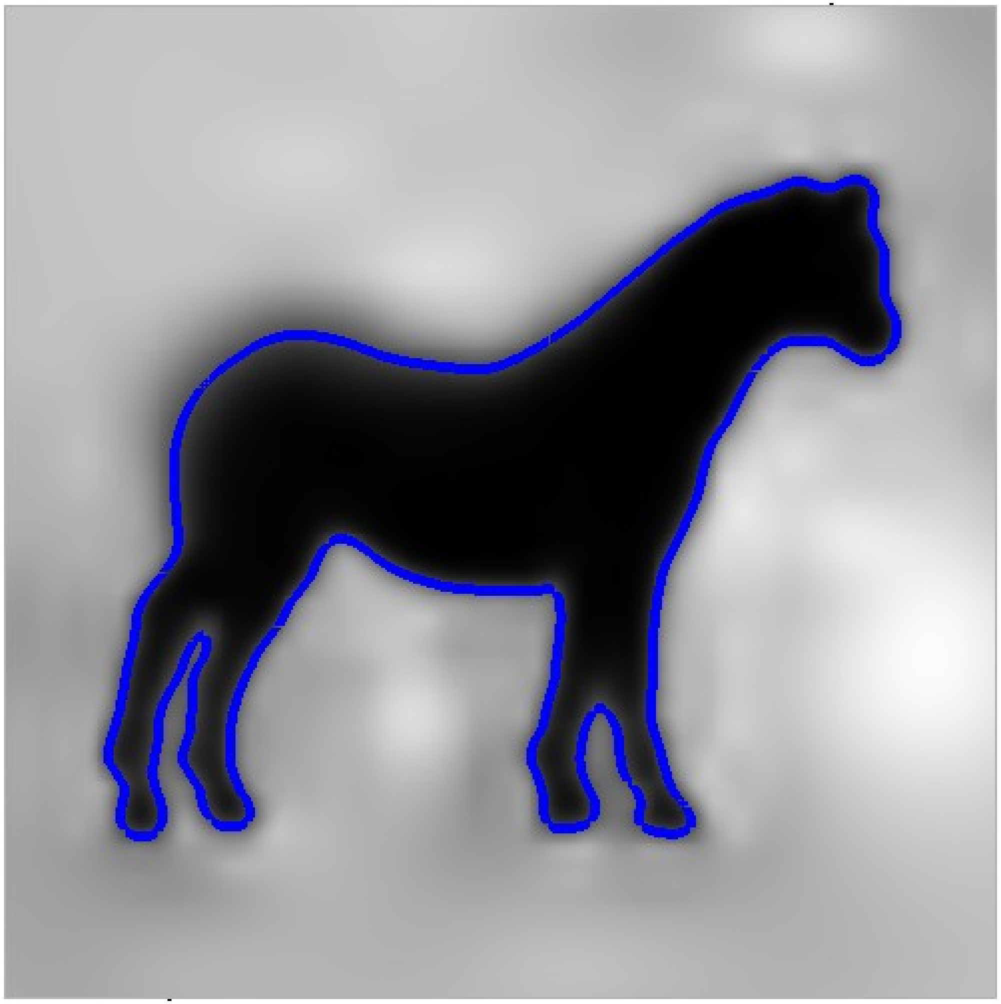


Implementation: Surface Extraction



Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
- Extract iso-surface