Distances on Surfaces

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A geodesic curve on a surface (technically, a Riemannian manifold) is a curve $y(t)$ such that:

- **Definition 1:** It describes the motion of a particle with acceleration along the surface normal $\ddot{y}(t) = c \hat{n}_{y(t)}$

  - Implies that geodesics have constant speed: $\| \dot{y}(t) \| = s$
A geodesic curve on a surface (technically, a Riemannian manifold) is a curve $y(t)$ such that:

- **Definition 2**: It is locally length-minimizing:
  - Around any point $y(t)$, there is a neighborhood $B_t = (t - \varepsilon, t - \varepsilon)$ such that the curve is the shortest path between any two points $p, q$ in $y(t \in B_t)$.
Geodesics ≠ Shortest Paths

• A geodesic is not necessarily the shortest path between two points
• ... but the shortest path is always a geodesic
But in common usage...

- ... we often use “geodesic” and “shortest path” interchangeably (and hence inaccurately)
- The shortest path between two points on a mesh is approximated by the distance along the edge graph
Existence and Uniqueness

- (Roughly) On a smooth manifold surface, if we're given a point $p$ and a vector $v$ in the tangent plane at $p$, then there is exactly one geodesic through $p$, with direction (tangent) $v$

- There can be multiple geodesics through the same point, for different $v$
Robot assists in activity by opening fridge door

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Average path lengths

- 225k Film actors: 3.65
- 5k nodes on US power grid: 18.7
- 282 neurons of C. elegans: 2.65
- 721m Facebook users: 4.74

Watts and Strogatz, 1998; Facebook, 2011
If geometry tells us about distances, what do distances tell us about geometry?
Can a 2D ant on a 2D surface tell if it lives in a space of positive, negative or zero curvature?
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Can a person, in 3D?
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Can a person, in 3D?

Yes, by measuring distances!
$K_0 < 0$ 

$K_0 > 0$

$C(\rho) > 2\pi \rho$

$C(\rho) < 2\pi \rho$

Gemmer and Venkataramani, 2013
Sum of angles $> 180^\circ$

Sum of angles $< 180^\circ$
M. C. Escher, Circle Limit III
How long is the coastline?
2800km
The Koch snowflake
The Koch snowflake
The Koch snowflake
The Koch snowflake

In the limit: bounded area, unbounded perimeter