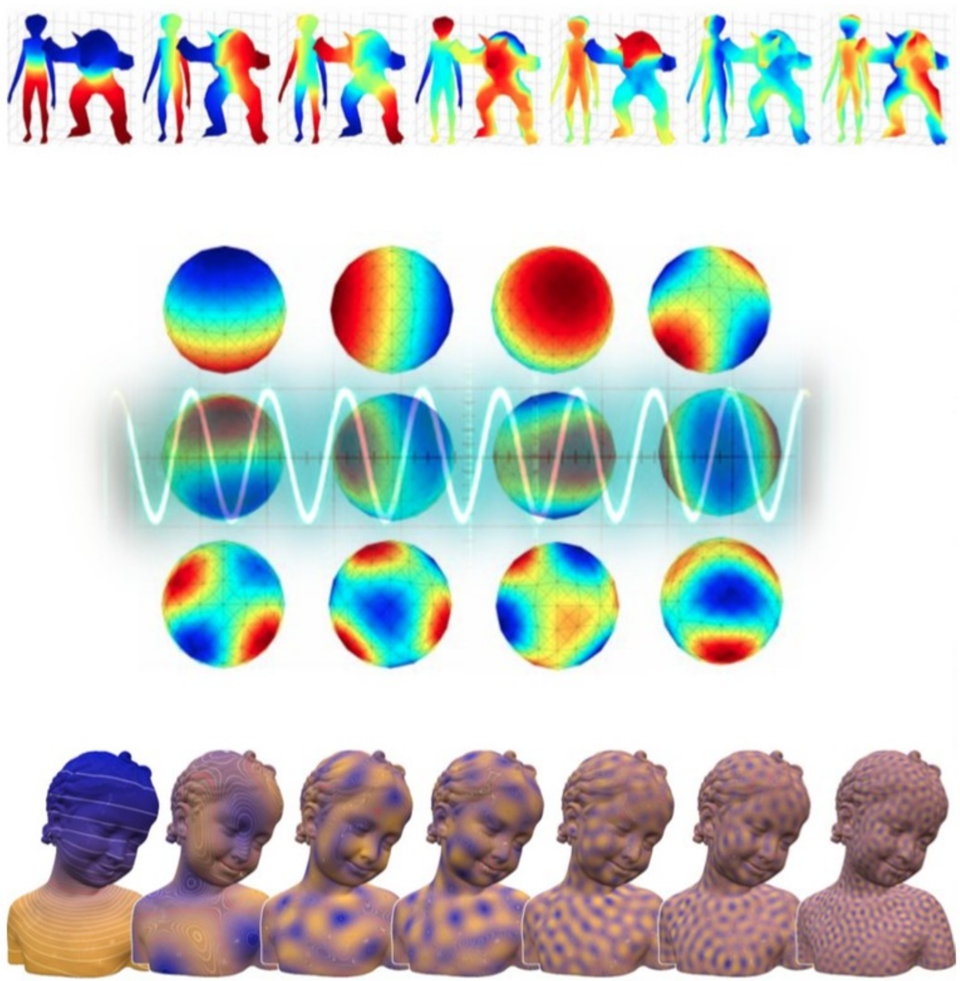
Levy and Zhang



Spectral Mesh Analysis

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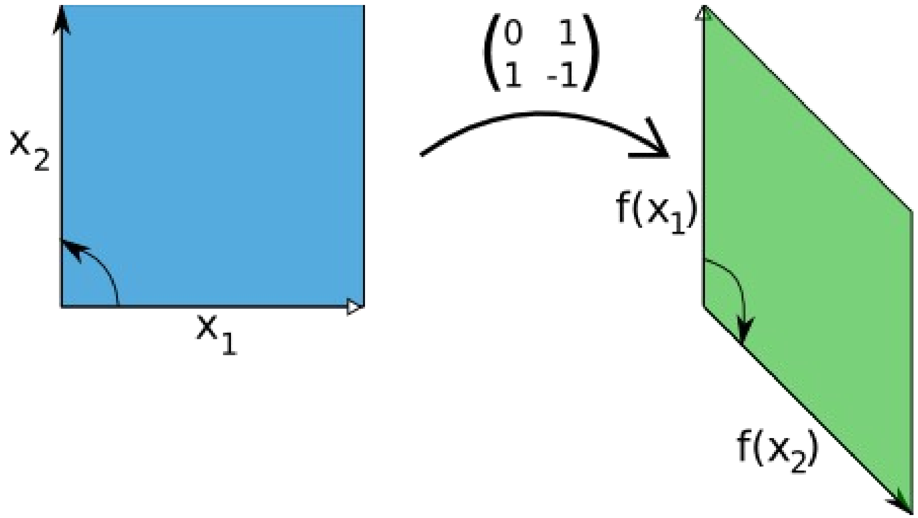
Matrices as transformations

- Let A be an $m \times n$ matrix
  - It can be thought of as a function that maps a vector $\mathbf{x} \in \mathbb{R}^n$ to a vector $A\mathbf{x} \in \mathbb{R}^m$
- A is a **linear transformation**
  - f is linear if $f(a + b) = f(a) + f(b)$

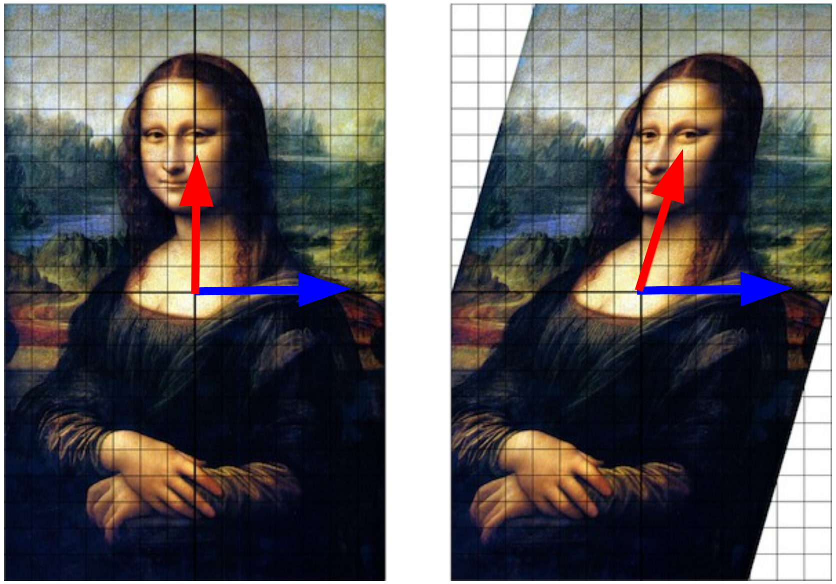


Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix
- An **eigenvalue** of A is a scalar λ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

  where $\mathbf{x}$ is some n-D vector
- $\mathbf{x}$ is the corresponding **eigenvector**
  - Interpretation: $\mathbf{x}$ is a vector that is left unchanged in direction by the linear transformation
  - It is *not unique*: $s\mathbf{x}$ is also an eigenvector for scalar s
  - If, for the same eigenvalue λ, there are k linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, then the eigenvalue is said to have **geometric multiplicity** k, and any linear combination $\Sigma_i\, w_i\, \mathbf{x}_i$ is also an eigenvector



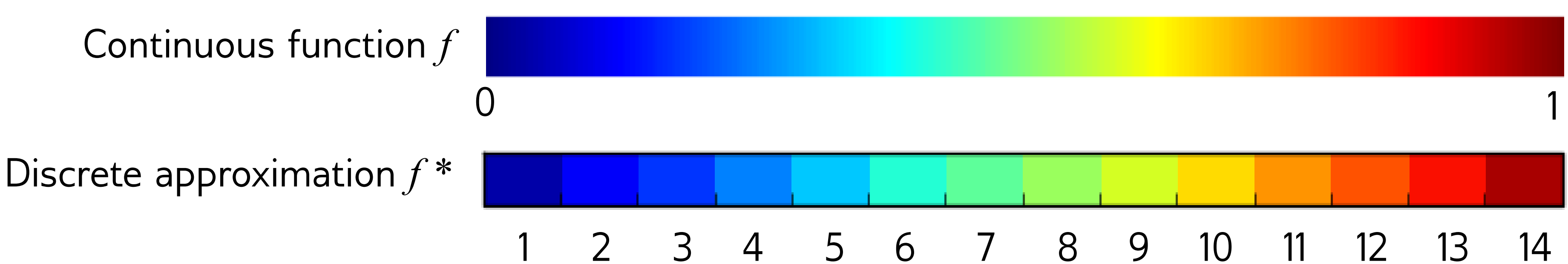
Blue arrow is eigenvector of shear transform, red is not

Functions as vectors

- Functions from A to B form a vector space: we can think of functions as "vectors"
  - E.g. we can commutatively add two functions: $f + g = g + f$
  - Or distribute multiplication with a scalar: $s(f + g) = sf + sg$
  - If we want, we can also associate a **norm** ("vector length") with a function: e.g. $\|f\| = \left(\int f^2(x)\,\mathrm{d}x\right)^{1/2}$

A function can be discretized

- Characterize a function f by its values at a finite set of n sample points
  - This results in a discrete function, let's call it f^*
  - The discrete function is perfectly defined by its values at the n points
  - In other words, f^* is represented by a finite-dimensional vector $[f(x_1), f(x_2), \ldots, f(x_n)]$



Linear operators

- An **operator** T is a mapping from a vector space U to another vector space V
  - T is a **linear operator** if $T(a + b) = T(a) + T(b)$
- The set of functions F from domain A to codomain B is a vector space
  - So we can have operators T that map from one function space F to another function space G
  - Note that T maps functions to functions!
- The differentials $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc are linear operators
  - They map functions to their derivatives

Eigenfunctions

- An **eigenvalue** of a linear operator T that maps a vector space to itself is a scalar λ s.t.

$$T(\mathbf{x}) = \lambda \mathbf{x}$$

  and $\mathbf{x}$ is the corresponding **eigenvector**

- If T maps functions to functions, then we call $\mathbf{x}$ an **eigenfunction**: $T(f) = \lambda f$

Discrete Linear Operators

- **Theorem:** Any linear operator between finite-dimensional vector spaces can be represented by a matrix
  - Let's say we have a set of functions F from A to B
  - The discrete versions of the functions form a finite-dimensional vector space F^* equivalent to $\mathbb{R}^n$
    - Each function is sampled at the same finite set of points
  - Let T be a linear operator from F to itself
  - … and T^* be a "discrete version" of T acting on F^*
  - Then T^* can be represented by a $n \times n$ matrix (cf. theorem)

Example: Discrete Derivative

Continuous

- Function: f
- Operator: $\dfrac{d}{dx}$
- Applying operator:

$$\frac{df}{dx} = f'$$

Discrete

- Vector: $\mathbf{f} = [\, f(x_1), f(x_2) \ldots f(x_n)]$
- Matrix:

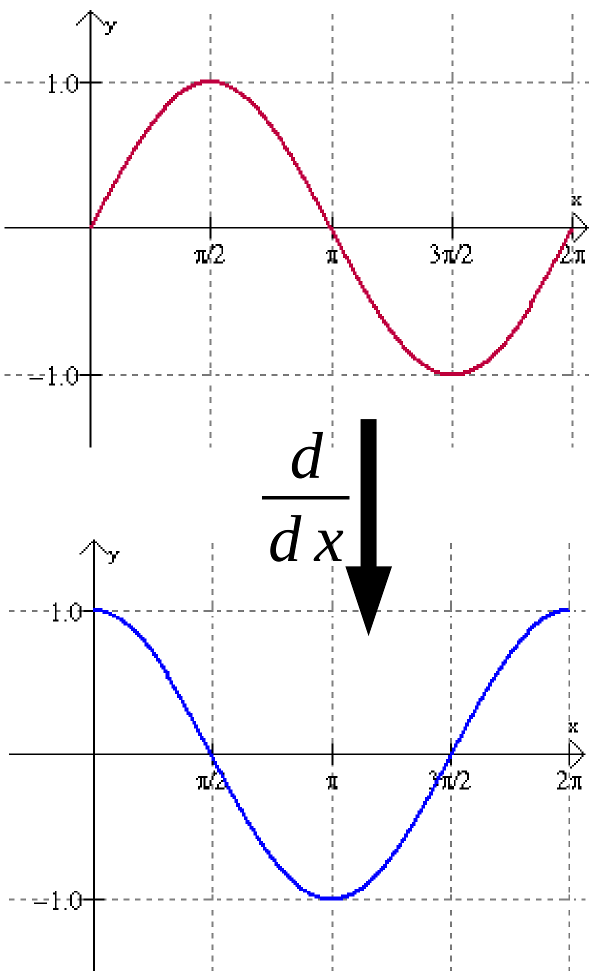
$$A = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ & 0 & -1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & -1 & 1 \\ 0 & 0 & \cdots & & 0 & -1 \end{bmatrix}$$

- Applying matrix:

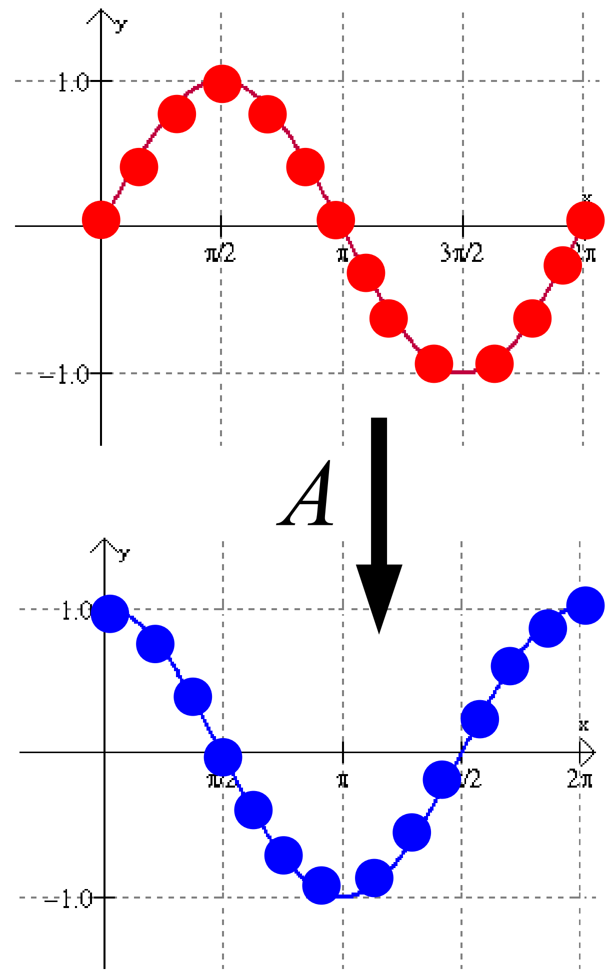
$$A\mathbf{f} = \mathbf{f}'$$

Example: Discrete Derivative

Continuous



Discrete



Example: Discrete 2nd Derivative

Continuous

- Function: f
- Operator: $\dfrac{d^2}{dx^2}$
- Applying operator:

$$\frac{d^2 f}{dx^2} = f''$$

Discrete

- Vector: $\mathbf{f} = [\, f(x_1), f(x_2) \ldots f(x_n)]$
- Matrix:

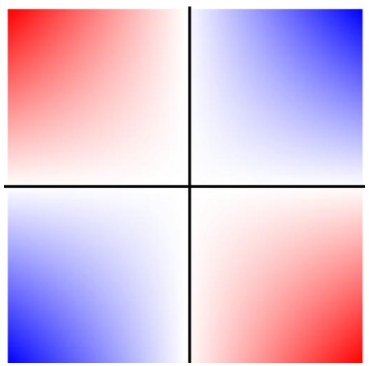
$$L = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ & 1 & -2 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & -2 & 1 \\ 0 & 0 & \cdots & & 1 & -2 \end{bmatrix}$$

- Applying matrix:

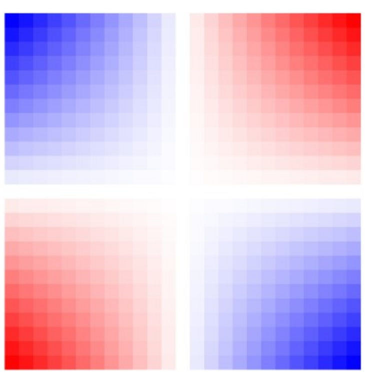
$$L\mathbf{f} = \mathbf{f}''$$

Operators in higher dimensions

- The underlying function space can have a higher-dimensional domain



Continuous function



Discrete approximation

$$\begin{bmatrix} -4 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -4 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -4 & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & -4 & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & -4 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -4 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & -4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & -4 \end{bmatrix}$$

2D discrete Laplace operator

Interpreting eigenfunctions

- Eigenvalues of a linear operator form its **spectrum**
- The eigenfunctions are unchanged (except for scaling) when transformed by the operator
  - Think of them as standing waves on the surface
- E.g.

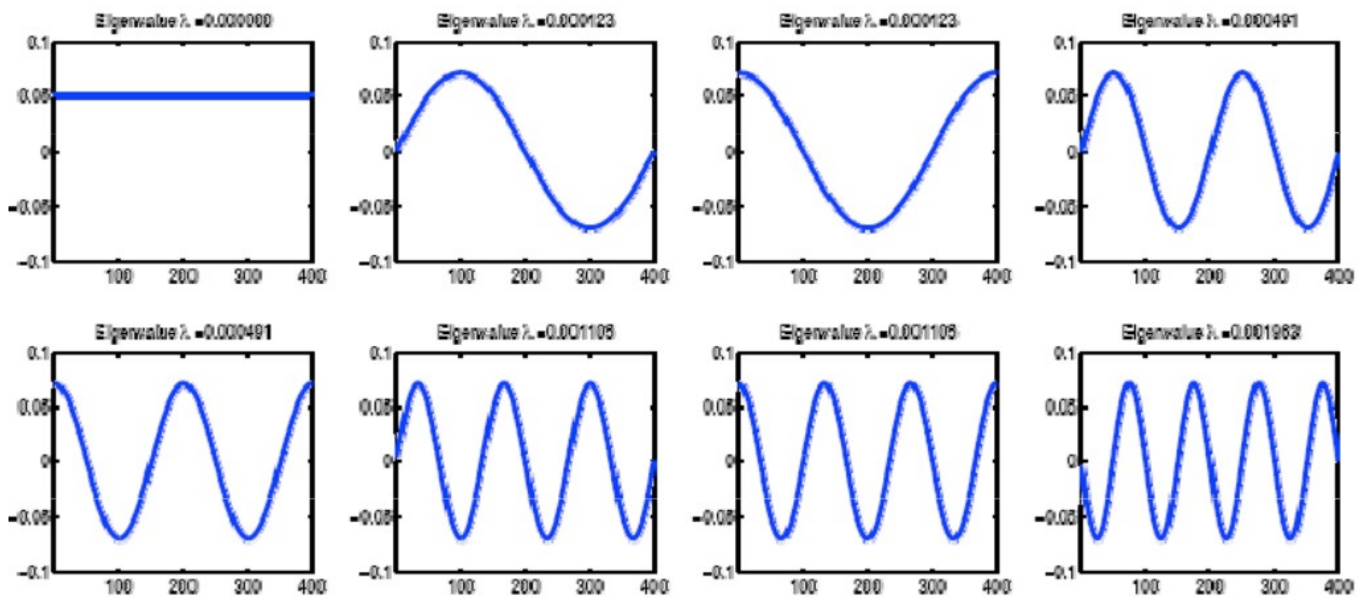
$$\frac{d^2 \sin(nx)}{dx^2} = -n^2 \times \sin(nx)$$

$$\frac{d^2 \cos(nx)}{dx^2} = -n^2 \times \cos(nx)$$

$$\frac{d^2 e^{\lambda x}}{dx^2} = \lambda^2 \times e^{\lambda x}$$

Interpreting eigenfunctions

- The eigenfunctions of the operator form a basis for the function space
  - E.g. the sinusoidal eigenfunctions of $\frac{d^2}{dx^2}$ form the Fourier basis



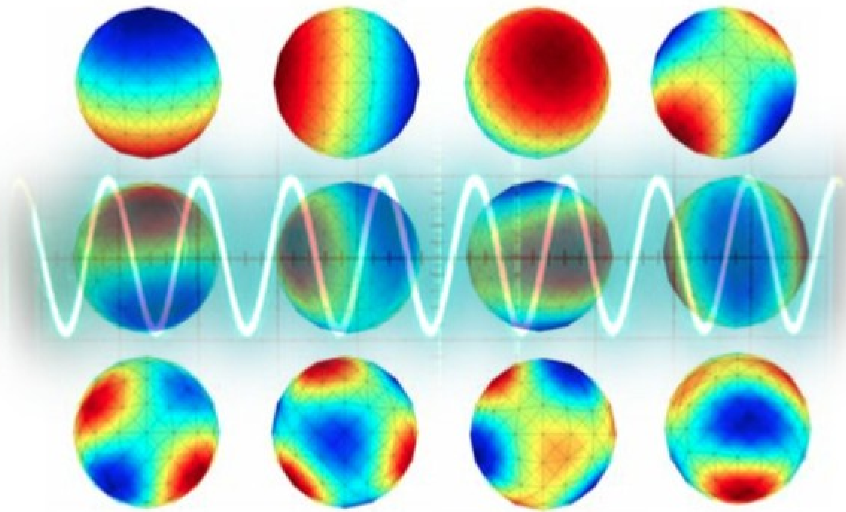
The first 8 sinusoidal eigenfunctions of the second derivative operator.
The eigenvalues are the negative squared frequencies.

Operators on manifolds

- We can define a function on a manifold curve/surface!
  - E.g. the coordinate function: gives the (X, Y, Z) position of a point on the surface
- A common operator is the Laplace-Beltrami operator
  - Its eigenfunctions define a basis for functions over the surface

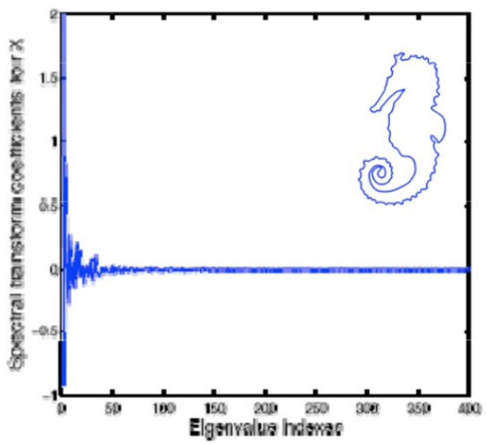
Eigenfunctions of Laplace-Beltrami

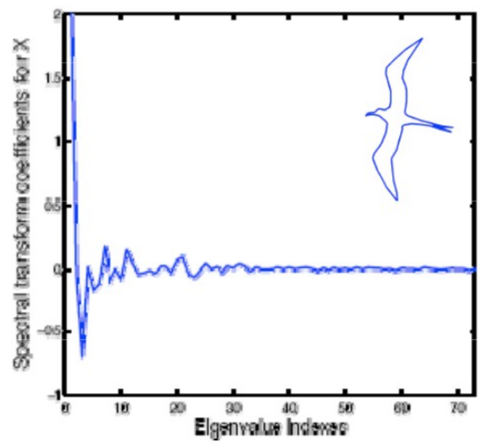
- *Intrinsic* basis for functions over surface
  - Doesn't change under isometry
- We can discretize it as usual: the function is defined at a fixed set of sample points on the shape



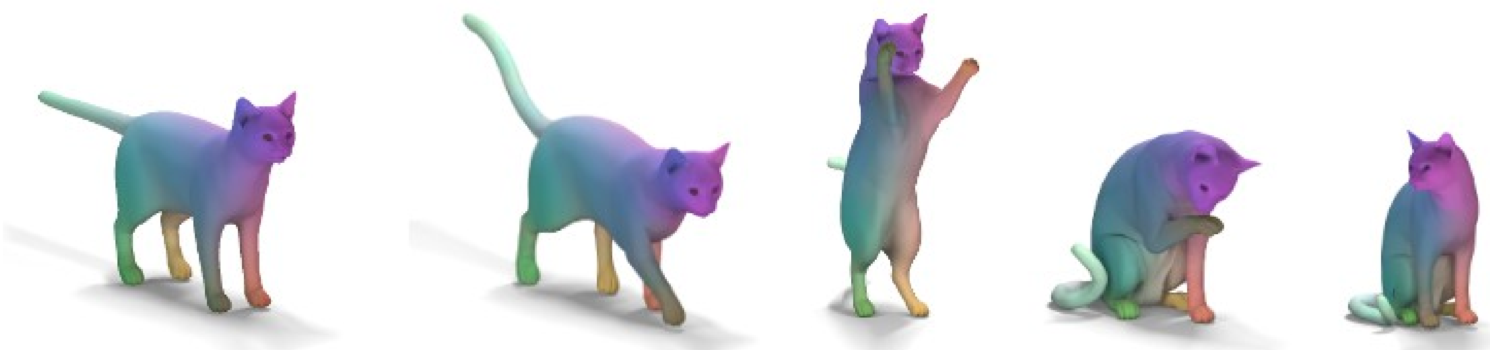
Eigenfunctions of Laplace-Beltrami

- The spectrum of the L-B operator characterizes the intrinsic geometry of the shape





- Two shapes related by isometry have the same Laplace-Beltrami spectrum



Expressing a function with eigenfunctions

- **Continuous:**

$$f(p) = w_1\,\varphi_1(p) + w_2\,\varphi_2(p) + \ldots + w_n\varphi_n(p)$$

- **Discrete:**

$$X = \sum_{i=1}^{n} \mathbf{e}_i \tilde{x}_i = \begin{bmatrix} E_{11} \\ E_{21} \\ \vdots \\ E_{n1} \end{bmatrix} \tilde{x}_1 + \ldots + \begin{bmatrix} E_{1n} \\ E_{2n} \\ \vdots \\ E_{nn} \end{bmatrix} \tilde{x}_n = \begin{bmatrix} E_{11} & \ldots & E_{1n} \\ E_{21} & \ldots & E_{2n} \\ \vdots & \vdots & \vdots \\ E_{n1} & \ldots & E_{nn} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = E\tilde{X}$$

$$\tilde{X} = E^{\mathrm{T}} X$$

$$\boxed{\tilde{x}_i = \mathbf{e}_i^{\mathrm{T}} \cdot X.}$$

Projection of *X* along eigenvector

The spectral transform

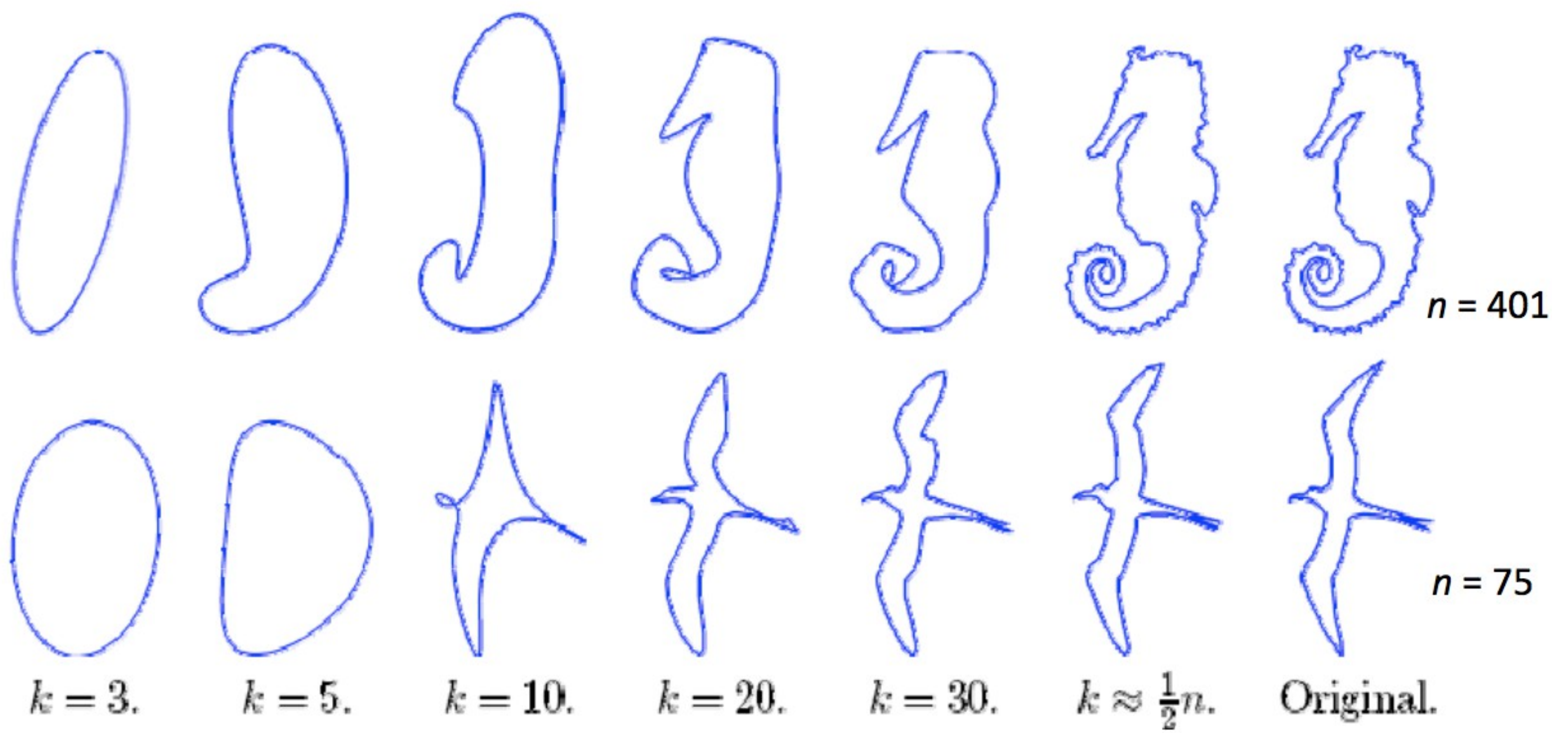
$$X \longrightarrow \tilde{X}$$

Spatial domain

Spectral domain

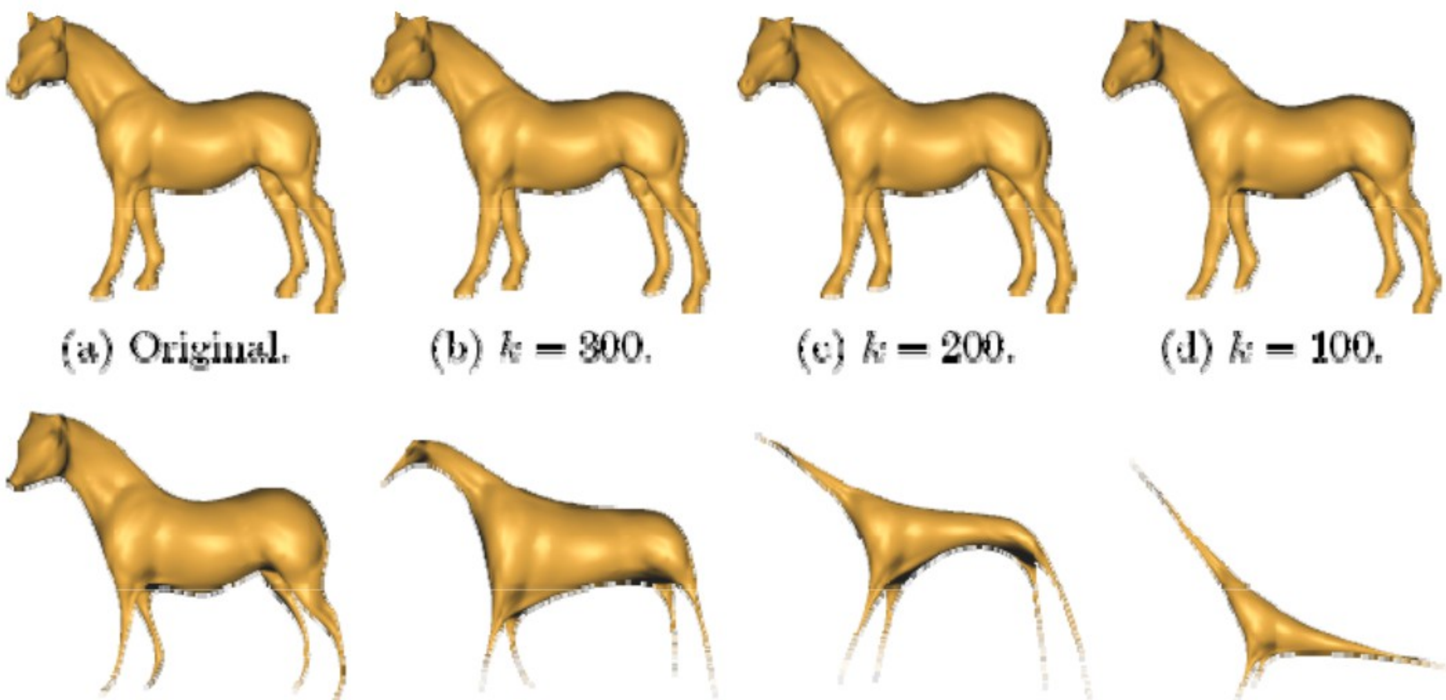
Reconstruction in 2D

- More accuracy with more eigenfunctions
- Function is the coordinate function
  - We're reconstructing the extrinsic shape of the object



Reconstruction in 3D

- More accuracy with more eigenfunctions
- Function is the coordinate function
  - We're reconstructing the extrinsic shape of the object



(a) Original. (b) $k = 300$. (c) $k = 200$. (d) $k = 100$.

(e) $k = 50$. (f) $k = 10$. (g) $k = 5$. (h) $k = 3$.