14.4. Normal Curvature and the Second Fundamental Form

In this section, we take a closer look at the curvature at a point of a curve C on a surface X.

Assuming that C is parameterized by arc length, we will see that the vector $X''(s)$ (which is equal to $\kappa \overrightarrow{n}$, where $\overrightarrow{n}$ is the principal normal to the curve C at p, and κ is the curvature) can be written as

$$\kappa \overrightarrow{n} = \kappa_N \mathbf{N} + \kappa_g \overrightarrow{n_g},$$

where $\mathbf{N}$ is the normal to the surface at p, and $\kappa_g \overrightarrow{n_g}$ is a tangential component normal to the curve.

The component κ_N is called the normal curvature.

Computing it will lead to the *second fundamental form*, another very important quadratic form associated with a surface.

The component κ_g is called the *geodesic curvature*.

It turns out that it only depends on the first fundamental form, but computing it is quite complicated, and this will lead to the *Christoffel symbols*.

Let $f\colon]a, b[\to \mathbb{E}^3$ be a curve, where f is a least C^3-continuous, and assume that the curve is parameterized by arc length.

We saw in Chapter 13, section 13.6, that if $f'(s) \neq 0$ and $f''(s) \neq 0$ for all $s \in]a, b[$ (i.e., f is biregular), we can associate to the point $f(s)$ an orthonormal frame $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ called the Frenet frame, where

$$\begin{aligned} \overrightarrow{t} &= f'(s), \\ \overrightarrow{n} &= \frac{f''(s)}{\|f''(s)\|}, \\ \overrightarrow{b} &= \overrightarrow{t} \times \overrightarrow{n}. \end{aligned}$$

The vector $\overrightarrow{t}$ is the unit *tangent vector*, the vector $\overrightarrow{n}$ is called the *principal normal*, and the vector $\overrightarrow{b}$ is called the *binormal*.

Furthermore the curvature κ at s is $\kappa = \|f''(s)\|$, and thus,

$$f''(s) = \kappa \overrightarrow{n}.$$

The principal normal $\overrightarrow{n}$ is contained in the osculating plane at s, which is just the plane spanned by $f'(s)$ and $f''(s)$.

Recall that since f is parameterized by arc length, the vector $f'(s)$ is a unit vector, and thus

$$f'(s) \cdot f''(s) = 0,$$

which shows that $f'(s)$ and $f''(s)$ are linearly independent and orthogonal, provided that $f'(s) \neq 0$ and $f''(s) \neq 0$.

Now, if $C\colon t \mapsto X(u(t), v(t))$ is a curve on a surface X, assuming that C is parameterized by arc length, which implies that

$$(s')^2 = E(u')^2 + 2Fu'v' + G(v')^2 = 1,$$

we have

$$X'(s) = X_u u' + X_v v',$$
$$X''(s) = \kappa \overrightarrow{n},$$

and $\overrightarrow{t} = X_u u' + X_v v'$ is indeed a unit tangent vector to the curve and to the surface, but $\overrightarrow{n}$ is the principal normal to the curve, and thus it is **not** necessarily orthogonal to the tangent plane $T_p(X)$ at $p = X(u(t), v(t))$.

Thus, if we intend to study how the curvature κ varies as the curve C passing through p changes, the Frenet frame $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ associated with the curve C is not really adequate, since both $\overrightarrow{n}$ and $\overrightarrow{b}$ will vary with C (and $\overrightarrow{n}$ is undefined when $\kappa = 0$).

Thus, it is better to pick a frame associated with the normal to the surface at p, and we pick the frame $(\overrightarrow{t}, \overrightarrow{n_g}, \mathbf{N})$ defined as follows.:

Definition 14.4.1 Given a surface X, given any curve $C\colon t \mapsto X(u(t), v(t))$ on X, for any point p on X, the orthonormal frame $(\overrightarrow{t}, \overrightarrow{n_g}, \mathbf{N})$ is defined such that

$$\begin{aligned}\overrightarrow{t} &= X_u u' + X_v v',\\ \mathbf{N} &= \frac{X_u \times X_v}{\|X_u \times X_v\|},\\ \overrightarrow{n_g} &= \mathbf{N} \times \overrightarrow{t},\end{aligned}$$

where $\mathbf{N}$ is the normal vector to the surface X at p. The vector $\overrightarrow{n_g}$ is called the *geodesic normal vector* (for reasons that will become clear later).

For simplicity of notation, we will often drop arrows above vectors if no confusion may arise.

Observe that $\overrightarrow{n_g}$ is the unit normal vector to the curve C contained in the tangent space $T_p(X)$ at p.

If we use the frame $(\overrightarrow{t}, \overrightarrow{n_g}, \mathbf{N})$, we will see shortly that $X''(s) = \kappa \overrightarrow{n}$ can be written as

$$\kappa \overrightarrow{n} = \kappa_N \mathbf{N} + \kappa_g \overrightarrow{n_g}.$$

The component $\kappa_N \mathbf{N}$ is the orthogonal projection of $\kappa \overrightarrow{n}$ onto the normal direction $\mathbf{N}$, and for this reason κ_N is called the *normal curvature of C at p.*

The component $\kappa_g \overrightarrow{n_g}$ is the orthogonal projection of $\kappa \overrightarrow{n}$ onto the tangent space $T_p(X)$ at p.

We now show how to compute the normal curvature. This will uncover the second fundamental form.

Using the abbreviations

$$X_{uu} = \frac{\partial^2 X}{\partial u^2}, \quad X_{uv} = \frac{\partial^2 X}{\partial u \partial v}, \quad X_{vv} = \frac{\partial^2 X}{\partial v^2},$$

since $X' = X_u u' + X_v v'$, using the chain rule, we get

$$X'' = X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2 + X_u u'' + X_v v''.$$

In order to decompose $X'' = \kappa \overrightarrow{n}$ into its normal component (along $\mathbf{N}$) and its tangential component, we use a neat trick suggested by Eugenio Calabi.

Recall that

$$(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w} = (\overrightarrow{u} \cdot \overrightarrow{w})\overrightarrow{v} - (\overrightarrow{w} \cdot \overrightarrow{v})\overrightarrow{u}.$$

Using this identity, we have

$$\begin{aligned}(\mathbf{N} \times (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2) \times \mathbf{N} \\ = (\mathbf{N} \cdot \mathbf{N})(X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2) \\ - (\mathbf{N} \cdot (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2))\mathbf{N}.\end{aligned}$$

Since $\mathbf{N}$ is a unit vector, we have $\mathbf{N} \cdot \mathbf{N} = 1$, and consequently, since

$$\kappa \overrightarrow{n} = X'' = X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2 + X_u u'' + X_v v'',$$

we can write

$$\begin{aligned}\kappa \overrightarrow{n} = (\mathbf{N} \cdot (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2))\mathbf{N} \\ + (\mathbf{N} \times (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2)) \times \mathbf{N} \\ + X_u u'' + X_v v''.\end{aligned}$$

Thus, it is clear that the normal component is

$$\kappa_N \mathbf{N} = (\mathbf{N} \cdot (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2))\mathbf{N},$$

and the normal curvature is given by

$$\kappa_N = \mathbf{N} \cdot (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2).$$

Letting

$$L = \mathbf{N} \cdot X_{uu}, \quad M = \mathbf{N} \cdot X_{uv}, \quad N = \mathbf{N} \cdot X_{vv},$$

we have

$$\kappa_N = L(u')^2 + 2Mu'v' + N(v')^2.$$

It should be noted that some authors (such as do Carmo) use the notation

$$e = \mathbf{N} \cdot X_{uu}, \quad f = \mathbf{N} \cdot X_{uv}, \quad g = \mathbf{N} \cdot X_{vv}.$$

Recalling that

$$\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|},$$

using the Lagrange identity

$$(\overrightarrow{u} \cdot \overrightarrow{v})^2 + \|\overrightarrow{u} \times \overrightarrow{v}\|^2 = \|\overrightarrow{u}\|^2 \|\overrightarrow{v}\|^2,$$

we see that

$$\|X_u \times X_v\| = \sqrt{EG - F^2},$$

and $L = \mathbf{N} \cdot X_{uu}$ can be written as

$$L = \frac{(X_u \times X_v) \cdot X_{uu}}{\sqrt{EG - F^2}} = \frac{(X_u, X_v, X_{uu})}{\sqrt{EG - F^2}},$$

where (X_u, X_v, X_{uu}) is the determinant of the three vectors.

Some authors (including Gauss himself and Darboux) use the notation

$$\begin{aligned} D &= (X_u, X_v, X_{uu}), \\ D' &= (X_u, X_v, X_{uv}), \\ D'' &= (X_u, X_v, X_{vv}), \end{aligned}$$

and we also have

$$L = \frac{D}{\sqrt{EG - F^2}}, \; M = \frac{D'}{\sqrt{EG - F^2}}, \; N = \frac{D''}{\sqrt{EG - F^2}}.$$

These expressions were used by Gauss to prove his famous *Theorema Egregium*.

Since the quadratic form $(x, y) \mapsto Lx^2 + 2Mxy + Ny^2$ plays a very important role in the theory of surfaces, we introduce the following definition.

Definition 14.4.2 Given a surface X, for any point $p = X(u, v)$ on X, letting

$$L = \mathbf{N} \cdot X_{uu}, \quad M = \mathbf{N} \cdot X_{uv}, \quad N = \mathbf{N} \cdot X_{vv},$$

where $\mathbf{N}$ is the unit normal at p, the quadratic form $(x, y) \mapsto Lx^2 + 2Mxy + Ny^2$ is called the *second fundamental form of* X *at* p. It is often denoted as II_p. For a curve C on the surface X (parameterized by arc length), the quantity κ_N given by the formula

$$\kappa_N = L(u')^2 + 2Mu'v' + N(v')^2$$

is called the *normal curvature of* C *at* p.

The second fundamental form was introduced by Gauss in 1827.

Unlike the first fundamental form, the second fundamental form is not necessarily positive or definite.

Properties of the surface expressible in terms of the first fundamental are called *intrinsic properties* of the surface X.

Properties of the surface expressible in terms of the second fundamental form are called *extrinsic properties* of the surface X. They have to do with the way the surface is immersed in $\mathbb{E}^3$.

As we shall see later, certain notions that appear to be extrinsic turn out to be intrinsic, such as the geodesic curvature and the Gaussian curvature.

This is another testimony to the genius of Gauss (and Bonnet, Christoffel, etc.).

Remark: As in the previous section, if X is not injective, the second fundamental form II_p is not well defined. Again, we will not worry too much about this, or assume X injective.

It should also be mentioned that the fact that the normal curvature is expressed as

$$\kappa_N = L(u')^2 + 2Mu'v' + N(v')^2$$

has the following immediate corollary known as *Meusnier's theorem* (1776).

Lemma 14.4.3 *All curves on a surface X and having the same tangent line at a given point $p \in X$ have the same normal curvature at p.*

In particular, if we consider the curves obtained by intersecting the surface with planes containing the normal at p, curves called *normal sections*, all curves tangent to a normal section at p have the same normal curvature as the normal section.

Furthermore, the principal normal of a normal section is collinear with the normal to the surface, and thus, $|\kappa| = |\kappa_N|$, where κ is the curvature of the normal section, and κ_N is the normal curvature of the normal section.

We will see in a later section how the curvature of normal sections varies.

We can easily give an expression for κ_N for an arbitrary parameterization.

Indeed, remember that

$$\left(\frac{ds}{dt}\right)^2 = \|\dot{C}\|^2 = E\,\dot{u}^2 + 2F\,\dot{u}\dot{v} + G\,\dot{v}^2,$$

and by the chain rule

$$u' = \frac{du}{ds} = \frac{du}{dt}\frac{dt}{ds},$$

and since a change of parameter is a diffeomorphism, we get

$$u' = \frac{\dot{u}}{\left(\frac{ds}{dt}\right)}$$

and from

$$\kappa_N = L(u')^2 + 2Mu'v' + N(v')^2,$$

we get

$$\kappa_N = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}.$$

It is remarkable that this expression of the normal curvature uses both the first and the second fundamental form!

We still need to compute the tangential part X_t'' of X''.

We found that the tangential part of X'' is

$$X''_t = (\mathbf{N} \times (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2)) \times \mathbf{N} \\ + X_u u'' + X_v v''.$$

This vector is clearly in the tangent space $T_p(X)$ (since the first part is orthogonal to $\mathbf{N}$, which is orthogonal to the tangent space).

Furthermore, X'' is orthogonal to X' (since $X' \cdot X' = 1$), and by dotting $X'' = \kappa_N \mathbf{N} + X''_t$ with $\overrightarrow{t} = X'$, since the component $\kappa_N \mathbf{N} \cdot \overrightarrow{t}$ is zero, we have $X''_t \cdot \overrightarrow{t} = 0$, and thus X''_t is also orthogonal to $\overrightarrow{t}$, which means that it is collinear with $\overrightarrow{n_g} = \mathbf{N} \times \overrightarrow{t}$.

Therefore, we showed that

$$\kappa \overrightarrow{n} = \kappa_N \mathbf{N} + \kappa_g \overrightarrow{n_g},$$

where

$$\kappa_N = L(u')^2 + 2Mu'v' + N(v')^2$$

and

$$\kappa_g \overrightarrow{n_g} = (\mathbf{N} \times (X_{uu}(u')^2 + 2X_{uv}u'v' + X_{vv}(v')^2)) \times \mathbf{N} + X_u u'' + X_v v''.$$

The term $\kappa_g \overrightarrow{n_g}$ is worth an official definition.

Definition 14.4.4 Given a surface X, given any curve $C\colon t \mapsto X(u(t), v(t))$ on X, for any point p on X, the quantity κ_g appearing in the expression

$$\kappa\overrightarrow{n} = \kappa_N \mathbf{N} + \kappa_g \overrightarrow{n_g}$$

giving the acceleration vector of X at p is called the *geodesic curvature of* C *at* p.

In the next section, we give an expression for $\kappa_g \overrightarrow{n_g}$ in terms of the basis (X_u, X_v).

14.5. Geodesic Curvature and the Christoffel Symbols

We showed that the tangential part of the curvature of a curve C on a surface is of the form $\kappa_g \overrightarrow{n_g}$.

We now show that κ_n can be computed only in terms of the first fundamental form of X, a result first proved by Ossian Bonnet circa 1848.

The computation is a bit involved, and it will lead us to the Christoffel symbols, introduced in 1869.

Since $\overrightarrow{n_g}$ is in the tangent space $T_p(X)$, and since (X_u, X_v) is a basis of $T_p(X)$, we can write

$$\kappa_g \overrightarrow{n_g} = AX_u + BX_v,$$

form some $A, B \in \mathbb{R}$.

However,

$$\kappa \overrightarrow{n} = \kappa_N \mathbf{N} + \kappa_g \overrightarrow{n_g},$$

and since $\mathbf{N}$ is normal to the tangent space, $\mathbf{N} \cdot X_u = \mathbf{N} \cdot X_v = 0$, and by dotting

$$\kappa_g \overrightarrow{n_g} = AX_u + BX_v$$

with X_u and X_v, since $E = X_u \cdot X_u$, $F = X_u \cdot X_v$, and $G = X_v \cdot X_v$, we get the equations:

$$\kappa \overrightarrow{n} \cdot X_u = EA + FB,$$
$$\kappa \overrightarrow{n} \cdot X_v = FA + GB.$$

On the other hand,

$$\kappa \overrightarrow{n} = X'' = X_u u'' + X_v v'' + X_{uu}(u')^2 + 2X_{uv} u'v' + X_{vv}(v')^2.$$

Dotting with X_u and X_v, we get

$$\begin{aligned}\kappa \overrightarrow{n} \cdot X_u &= Eu'' + Fv'' + (X_{uu} \cdot X_u)(u')^2 \\ &\qquad + 2(X_{uv} \cdot X_u)u'v' + (X_{vv} \cdot X_u)(v')^2, \\ \kappa \overrightarrow{n} \cdot X_v &= Fu'' + Gv'' + (X_{uu} \cdot X_v)(u')^2 \\ &\qquad + 2(X_{uv} \cdot X_v)u'v' + (X_{vv} \cdot X_v)(v')^2.\end{aligned}$$

At this point, it is useful to introduce the *Christoffel symbols (of the first kind)* $[\alpha\, \beta;\, \gamma]$, defined such that

$$[\alpha\, \beta;\, \gamma] = X_{\alpha\beta} \cdot X_\gamma,$$

where $\alpha, \beta, \gamma \in \{u, v\}$. It is also more convenient to let $u = u_1$ and $v = u_2$, and to denote $[u_\alpha\, v_\beta;\, u_\gamma]$ as $[\alpha\, \beta;\, \gamma]$.

Doing so, and remembering that

$$\kappa \overrightarrow{n} \cdot X_u = EA + FB,$$
$$\kappa \overrightarrow{n} \cdot X_v = FA + GB,$$

we have the following equation:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_1'' \\ u_2'' \end{pmatrix} + \sum_{\substack{\alpha=1,2 \\ \beta=1,2}} \begin{pmatrix} [\alpha\,\beta;\, 1]\, u'_\alpha u'_\beta \\ [\alpha\,\beta;\, 2]\, u'_\alpha u'_\beta \end{pmatrix}.$$

However, since the first fundamental form is positive definite, $EG - F^2 > 0$, and we have

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = (EG - F^2)^{-1} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

Thus, we get

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} u_1'' \\ u_2'' \end{pmatrix} + \sum_{\substack{\alpha=1,2 \\ \beta=1,2}} (EG - F^2)^{-1} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} [\alpha\, \beta;\, 1]\, u_\alpha' u_\beta' \\ [\alpha\, \beta;\, 2]\, u_\alpha' u_\beta' \end{pmatrix}.$$

It is natural to introduce the *Christoffel symbols (of the second kind)* $\Gamma^k_{i\,j}$, defined such that

$$\begin{pmatrix} \Gamma^1_{i\,j} \\ \Gamma^2_{i\,j} \end{pmatrix} = (EG - F^2)^{-1} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} [i\, j;\, 1] \\ [i\, j;\, 2] \end{pmatrix}.$$

Finally, we get

$$A = u_1'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^{1}\, u_i' u_j',$$

$$B = u_2'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^{2}\, u_i' u_j',$$

and

$$\kappa_g \overrightarrow{n_g} =$$

$$\left(u_1'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^{1}\, u_i' u_j' \right) X_u + \left(u_2'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^{2}\, u_i' u_j' \right) X_v.$$

We summarize all the above in the following lemma.

Lemma 14.5.1 *Given a surface X and a curve C on X, for any point p on C, the tangential part of the curvature at p is given by*

$$\kappa_g \overrightarrow{n_g} =$$

$$\left(u_1'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^1 \, u_i' u_j' \right) X_u + \left(u_2'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^2 \, u_i' u_j' \right) X_v,$$

where the Christoffel symbols $\Gamma_{i\,j}^k$ are defined such that

$$\begin{pmatrix} \Gamma_{i\,j}^1 \\ \Gamma_{i\,j}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} [i\,j;\,1] \\ [i\,j;\,2] \end{pmatrix},$$

and the Christoffel symbols $[i\,j;\,k]$ are defined such that

$$[i\,j;\,k] = X_{ij} \cdot X_k.$$

Note that

$$[i\,j;\,k] = [j\,i;\,k] = X_{ij} \cdot X_k.$$

Looking at the formulae

$$[\alpha\,\beta;\ \gamma] = X_{\alpha\beta}\cdot X_{\gamma}$$

for the Christoffel symbols $[\alpha\,\beta;\ \gamma]$, it does not seem that these symbols only depend on the first fundamental form, but in fact they do!

After some calculations, we have the following formulae showing that the Christoffel symbols only depend on the first fundamental form:

$$\begin{aligned}
[1\,1;\ 1] &= \frac{1}{2}E_u, \quad [1\,1;\ 2] = F_u - \frac{1}{2}E_v,\\
[1\,2;\ 1] &= \frac{1}{2}E_v, \quad [1\,2;\ 2] = \frac{1}{2}G_u,\\
[2\,1;\ 1] &= \frac{1}{2}E_v, \quad [2\,1;\ 2] = \frac{1}{2}G_u,\\
[2\,2;\ 1] &= F_v - \frac{1}{2}G_u, \quad [2\,2;\ 2] = \frac{1}{2}G_v.
\end{aligned}$$

Another way to compute the Christoffel symbols $[\alpha\,\beta;\,\gamma]$, is to proceed as follows. For this computation, it is more convenient to assume that $u = u_1$ and $v = u_2$, and that the first fundamental form is expressed by the matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where $g_{\alpha\beta} = X_\alpha \cdot X_\beta$. Let

$$g_{\alpha\beta|\gamma} = \frac{\partial g_{\alpha\beta}}{\partial u_\gamma}.$$

Then, we have

$$g_{\alpha\beta|\gamma} = \frac{\partial g_{\alpha\beta}}{\partial u_\gamma} = X_{\alpha\gamma} \cdot X_\beta + X_\alpha \cdot X_{\beta\gamma} = [\alpha\,\gamma;\,\beta] + [\beta\,\gamma;\,\alpha].$$

From this, we also have

$$g_{\beta\gamma|\alpha} = [\alpha\,\beta;\,\gamma] + [\alpha\,\gamma;\,\beta],$$

and

$$g_{\alpha\gamma|\beta} = [\alpha\,\beta;\,\gamma] + [\beta\,\gamma;\,\alpha].$$

From all this, we get

$$2[\alpha\,\beta;\,\gamma] = g_{\alpha\gamma|\beta} + g_{\beta\gamma|\alpha} - g_{\alpha\beta|\gamma}.$$

As before, the Christoffel symbols $[\alpha\,\beta;\,\gamma]$ and $\Gamma^{\gamma}_{\alpha\,\beta}$ are related via the Riemannian metric by the equations

$$\Gamma^{\gamma}_{\alpha\,\beta} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} [\alpha\,\beta;\,\gamma].$$

This seemingly bizarre approach has the advantage to generalize to Riemannian manifolds. In the next section, we study the variation of the normal curvature.