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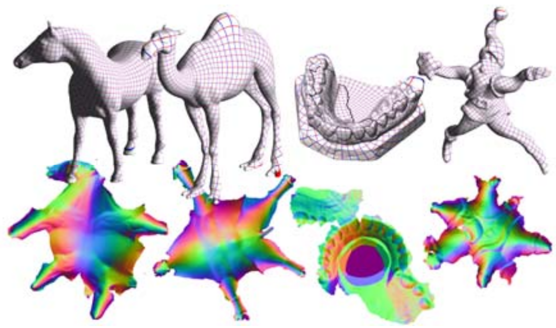
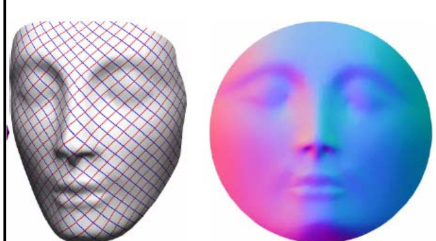
Parameterization – Free Boundary

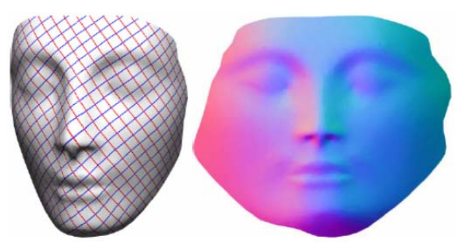


Free Boundary Methods

- Direct energy minimization
  - Example: Least Squares Conformal Map (LSCM)….
- Indirect
  - Example: Angle Based Flattening (ABF)….



ights [Floater 2003] on a circular domain.



LSCM [Lévy et al. 2002].

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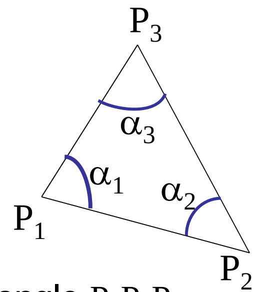
Free vs Fixed

LSCM – Geometric Interpretation



- Algebraic Interpretation:
  - Minimize conformal energy

$$E_C = (\sigma_1 - \sigma_2)^2/2$$

- Geometric Interpretation:
  - Use triangle similarity
  - Given angles $\alpha_1, \alpha_2, \alpha_3$ of a triangle $P_1P_2P_3$ in 2D we have

$$P_3 - P_1 = \frac{\sin\alpha_2}{\sin\alpha_3} R_{\alpha_1}(P_2 - P_1),$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

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LSCM

- In map from 3D to 2D might be impossible to keep angles exactly
  - Use least-squares

$$\min \sum_i \left(P_3^i - P^i{}_1 - \frac{\sin\alpha^i{}_2}{\sin\alpha^i{}_3} R_{\alpha^i{}_1}(P^i{}_2 - P_1^i)\right)^2$$

- To solve need to fix two vertices
  - Obtain linear system
  - Choice of vertices affects solution
- Can have flips

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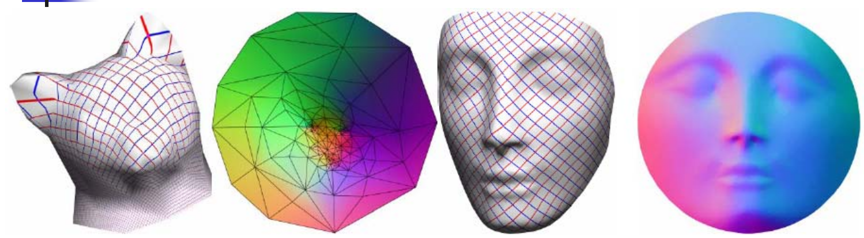
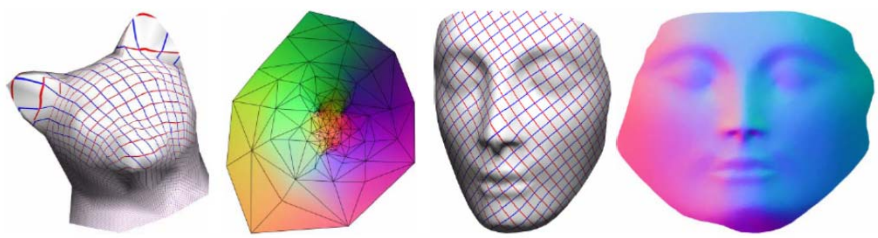
Examples



Parameterization with mean value weights [Floater 2003] on a circular domain.



Parameterization with LSCM [Lévy et al. 2002].

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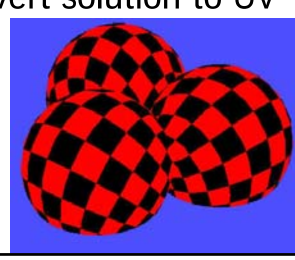
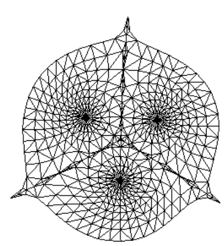
ABF: Angle Based Flattening

- Triangular 2D mesh is defined by its angles
- Formulate parameterization as problem in angle space
- Angle based formulation:
  - Distortion as function of angles (conformality)
  - Validity: set of angle constraints
  - Convert solution to UV





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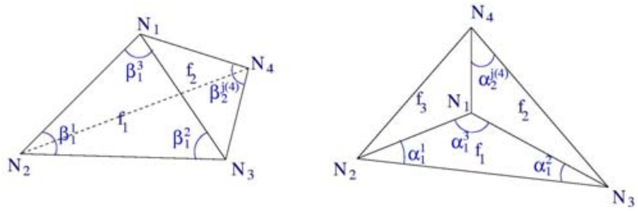
ABF Formulation



- Distortion:
  - 2D/3D angle difference

$$\sum_{t\in T,\, j=1..3} w_j^t\left(\alpha_j^t - \beta_j^t\right)^2,\ w_j^t = 1\big/{\beta_j^t}^2$$

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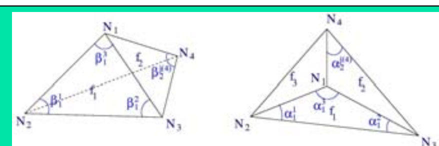
ABF Formulation



- Distortion:

$$\sum_{t\in T,\, j=1..3} w_j^t\left(\alpha_j^t - \beta_j^t\right)^2,\ w_j^t = 1\big/{\beta_j^t}^2$$

- ***Constraints:***
  - Triangle validity:

$$\forall t \in T, \quad \alpha_1^t + \alpha_2^t + \alpha_3^t - \pi = 0;$$

  - Planarity:

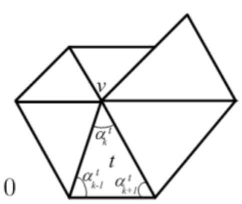
$$\forall v \in V_{int}, \quad \sum_{(t,k)\in v^*} \alpha_k^t - 2\pi = 0$$

  - Reconstruction

$$\forall v \in V_{int}, \quad \prod_{(t,k)\in v^*} \sin \alpha_{k\oplus 1}^t - \prod_{(t,k)\in v^*} \sin \alpha_{k\ominus 1}^t = 0$$

  - Positivity

$$\alpha_j^t > 0$$



- Solve - constrained optimization (Lagrange multipliers)

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Angle to UV Conversion

- Alternative 1: Use computed angles as input to LSCM
- Alternative 2: Unfolding
  - Choose one edge & place in 2D (keep length)
  - Based on computed angles place third vertex of triangles sharing this edge in 2D
    - Intersection of two rays
  - Repeat recursively

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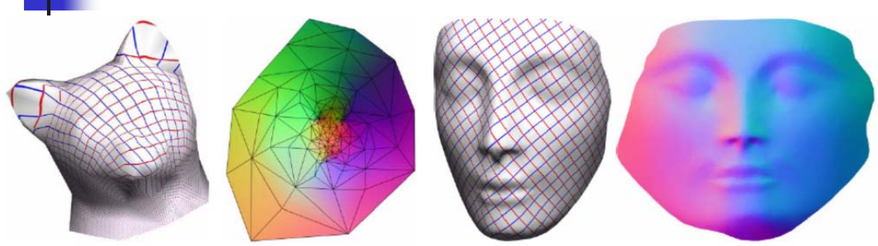
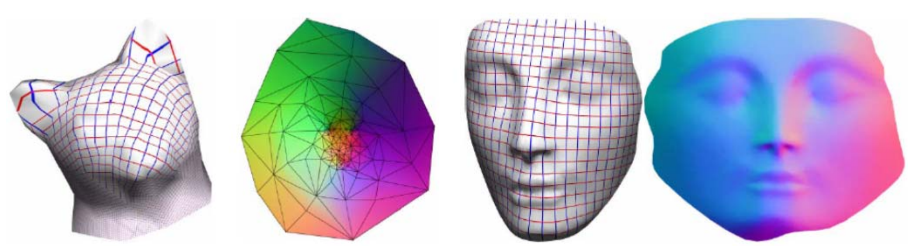
Examples



Parameterization with LSCM [Lévy et al. 2002].



Parameterization with ABF++ [Sheffer et al. 2005].

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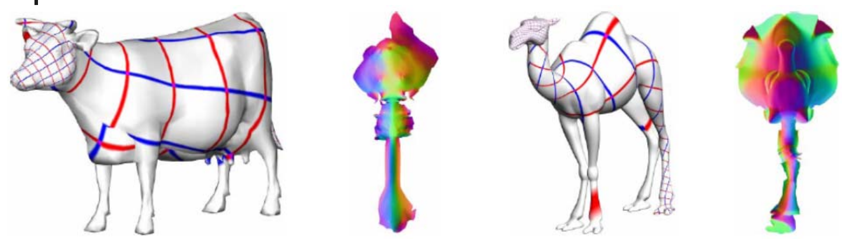
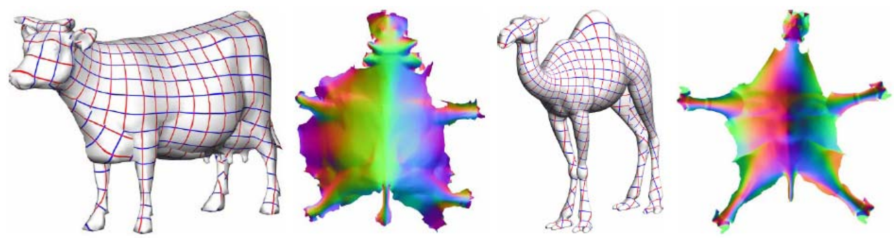
Examples



Parameterization with LSCM [Lévy et al. 2002].



Parameterization with ABF++ [Sheffer et al. 2005].

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2D Parameterization Summary

- Needed for many processing operations
- Distortion/Bijectivity important
- Trade-of quality/efficiency (as always…)
- Very popular topic (100+ major publications in last 10 years, less recently)
- More Issues
  - Segmentation/Cutting
  - Constraints

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Numerical Issues

Minimization with Constraints

- Need to
  - Find x such that $F(x)$ minimal
  - WHEN constraints $c(x) = 0$ satisfied
- Achieved when
  - $F'(x)=\mu c'(x)$
  - for unknown μ
- General formulation
  - $F^*(x,\mu)=F(x)+\mu c(x)$
  - Find x,μ which extremize F^*
  - Known as min-max
    - min on x
    - max on μ

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Solution

- Use Lagrange Multipliers

$$F^*(\alpha,\mu) = F(\alpha) + \mu_1 g^2(\alpha) + \mu_2 g^3(\alpha) + \mu_3 g^4(\alpha)$$

- Solve the min-max problem (minimum on α, maximum on μ)
- Reached when all derivatives are zero
- Have non-linear system of equations
- Use Newton method to solve

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Minimization (Unconstrained)

- To find x that minimizes $F(x)$ – find x such that $F'(x)=0$
  - Check if got minimum/maximum/saddle point
  - Note: finds **LOCAL** minimum
- Typically no need for explicit check (assume function does not have maxima/saddles
- Translate problem into: find x such that $f(x)=0$

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Solving Non-Linear Equations - Newton Method

- Consider Taylor expansion

$$f(x_0+\epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{1}{2}f''(x_0)\epsilon^2 + \ldots.$$

- Neglect terms > 1

$$f(x_0+\epsilon) \approx f(x_0) + f'(x_0)\epsilon.$$

- In 1D
  - Set x_0 – initial guess
  - While $f(x_i)$ not 0
    - $dx = -f(x_i)/f'(x_i)$
    - $x_{i+1} = x_i + dx$

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Newton Method in nD

- $f(x)$ vector
- $\Delta f(x)$ – matrix
- Set x_1 – initial guess
- While $||f(x_i)|| > 0$
  - Solve $\Delta f(x_i)\ dx = -f(x_i)$
    - Solve linear system
  - $x_{i+1} = x_i + dx$

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Solving Linear System

- Solve $Ax=B$ (A $n \times n$ matrix)
- Choice I: Compute A^{-1} $\boldsymbol{O(n^3)}$ TERRIBLY expensive
- Choice II: Iterative (Gauss/Gauss-Seidel)
  - Set x to initial guess
  - Solve one equation at a time
    - $A_i x=B_i$ - consider all x_j ($j \neq i$) as constant and compute x_i
      - $x_i = (b_i - \Sigma a_j x_j)/a_i$
    - Repeat (for all i) till convergence
  - Works only for a very small set of matrices

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Solving Linear System

- Choice III: LU (or LDL^T) decomposition
  - Compute matrices L & U such that
    - $LU=A$
    - L – lower matrix (has 1's on diagonal & 0's above)
    - U – upper matrix (has 0's below diagonal
    - Use off-the-shelf algorithm/code
      - Take advantage of sparsity (if applicable)
  - Solve:
    - Solve $Ly=B$ (use Gauss iterations)
      - Works (at each point add ONE variable)
    - Solve $Ux=y$ (use Gauss iterations)
      - Start from $i=n-1$ and go "up"
      - Works (at each point add ONE variable)

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