Differential Geometry
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Surfaces

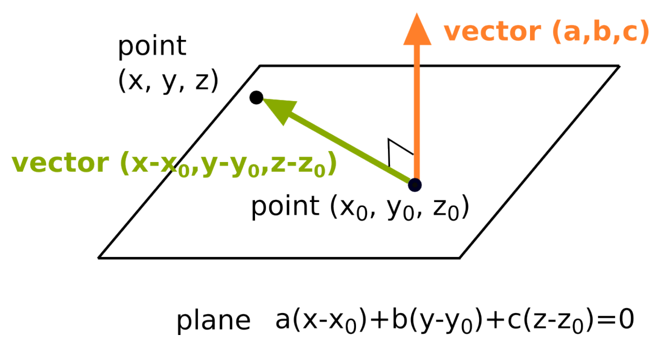
When studying curves, we studied how the curve twisted and turned in space. We now turn to surfaces, two-dimensional objects in three-dimensional space and examine how the concept of curvature translates to surfaces.

In Calculus 3, you have encounter surfaces defined as graphs of real valued functions of two variables $z = f(x, y)$. This function also can take the form $x = f(y, z)$ or $y = f(x, z)$. In some cases, on the other hand, this function is given implicitly as $F(x, y, z) = 0$. For example, a sphere of radius a is given by $x^2 + y^2 + z^2 = a^2$ and it is impossible to get a single two variable function that would describe the whole sphere. Cylinder $x^2 + y^2 = a^2$ is another such example. Let us review some examples from Calculus 3.

Planes. The general equation of a plane is

$$ax + by + cz + d = 0.$$

A plane is uniquely determined by a point in it and a vector perpendicular to it. The equation that describes any point $\mathbf{x} = (x, y, z)$ in the plane through a point $\mathbf{x}_0 = (x_0, y_0, z_0)$ perpendicular to a vector $\mathbf{a} = (a, b, c)$ is



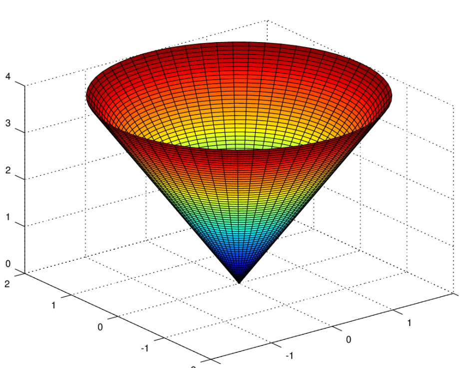
$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

The above vector equation of the plane has the following scalar form.

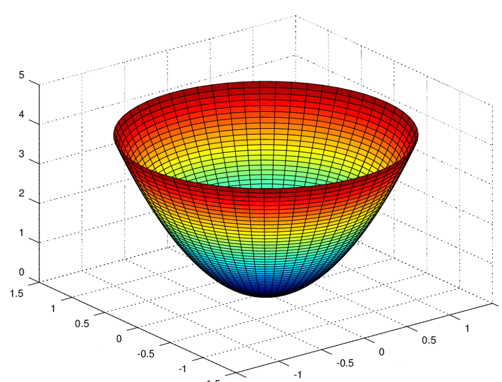
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Surfaces of revolution. $z = f(\sqrt{x^2 + y^2})$. To get graph of such surface, graph the function $z = f(y)$ in yz-plane and let it rotate about z-axis. For example

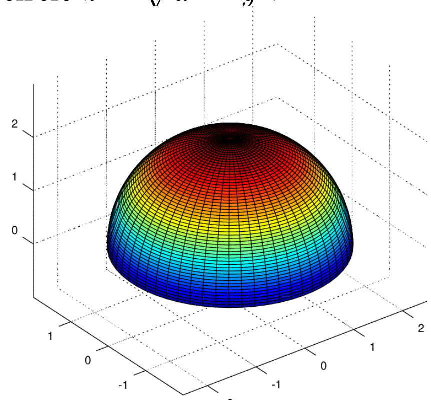
- A cone $z = a\sqrt{x^2 + y^2}$ is obtained by rotating the line $z = ay$.
- A paraboloid $z = ax^2 + ay^2$ is obtained by rotating the parabola $z = ay^2$.
- A half-sphere $z = \sqrt{a^2 - x^2 - y^2}$ is obtained by rotating the half-circle $z = \sqrt{a^2 - y^2}$.



Cone



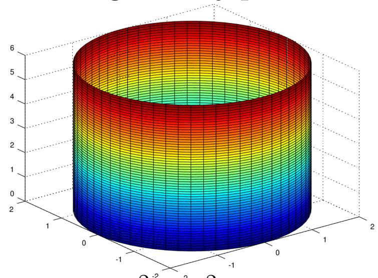
Paraboloid



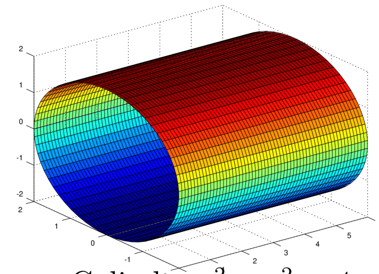
Hemisphere

Cylindrical surfaces. These surfaces are given by an equation in which one variable is not present. For example, $z = f(y)$. To graph this surface, graph the function $z = f(y)$ in yz-plane and translate the graph in direction of x-axis. For example,

- The graph of $y^2 + z^2 = 4$ is a cylinder obtained by translating the circle $y^2 + z^2 = 4$ of radius 2 centered at the origin in yz-plane along x-axis.
- The graph of $x^2 + y^2 = 4$ is a cylinder obtained by translating the circle $x^2 + y^2 = 4$ of radius 2 centered at the origin in xy-plane along z-axis.



Cylinder $x^2 + y^2 = 4$



Cylinder $y^2 + z^2 = 4$

Parametric Surfaces

In cases when a surface is given as an implicit function $F(x, y, z) = 0$, it may be useful to describe the three variables x, y and z but using some other parameters u and v. In that case, we have that

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v).$$

These equations are called **parametric equations** of the surface and the surface given via parametric equations is called a **parametric surface.**

Thus, a parametric surface is represented as a vector function of two variables, i.e. the **domain** D consisting of all possible values of parameters u and v is contained in $\mathbb{R}^2$. The **range** of the surfaces is contained in the three dimensional space $\mathbb{R}^3$.

Thus, a surface $\mathbf{x}$ is a mapping of D into $\mathbb{R}^3$. This is denoted by $\mathbf{x} : D \to \mathbb{R}^3$. The vector function $\mathbf{x}$ can also be represented as

$$\mathbf{x}(u, v) = (x(u, v),\ y(u, v),\ z(u, v)).$$

Notice an analogy with curves. We can think of curves as one-dimensional objects in three-dimensional space and surfaces as two-dimensional objects in three dimensional space. Thus, a curve can be described using a **single** parameter t. Surface, on the other hand, is described using **two** parameters u and v.

	Mapping	Dimension	Parameter(s)	Equations
Curve	$\boldsymbol{\gamma} : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^3$	1	t	$\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$
Surface	$\mathbf{x} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$	2	u, v	$\mathbf{x}(u, v) = (x(u, v),\ y(u, v),\ z(u, v))$

The curves given in the form $z = f(x, y)$ can always be parametrized as $\mathbf{x} = (x, y, f(x, y))$. For example, the plane $2x+3y+z = 6$ can be represented by $\mathbf{x} = (x, y, 6-2x-3y)$. Some other surfaces require fancier parametrizations. Recall the following change of coordinates used in Calculus 3 that provided parametric representations of some frequently used surfaces.

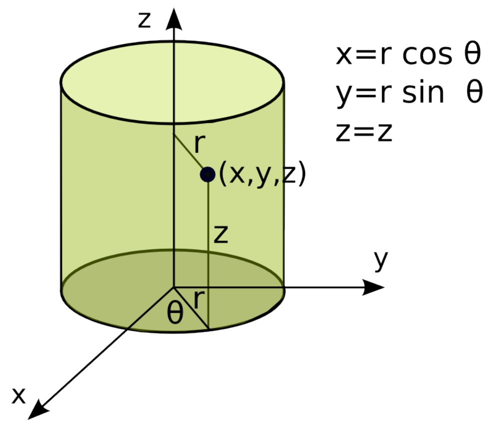
Cylindrical coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Here x and y are converted using polar coordinates and the only change in z may come just from changes in x and y. Note that

$$x^2 + y^2 = r^2$$

in these coordinates.



Using cylindrical coordinates we obtain parametrizations in the following examples.

1. The paraboloid $z = x^2 + y^2$ can be represented as $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2$, since $x^2 + y^2 = r^2$ in cylindrical coordinates so $z = r^2$. In this course, we write this parametrization shortly as
$$\mathbf{x} = (r\cos\theta, r\sin\theta, r^2).$$
Note that this paraboloid can also be parametrized by $(x, y, x^2 + y^2)$.

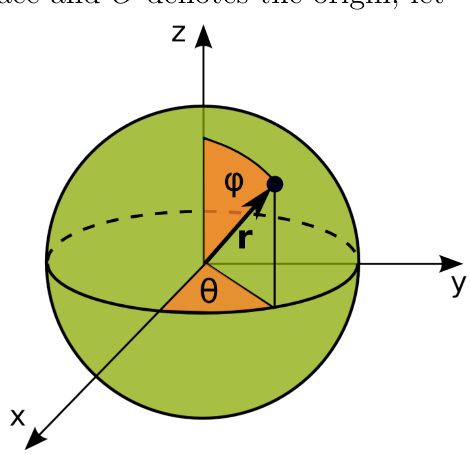
2. The cone $z = \sqrt{x^2 + y^2}$ can be represented by $\mathbf{x} = (r\cos\theta, r\sin\theta, r)$.

3. The cylinder $x^2 + y^2 = 4$ is such that the r value is constant and equal to 2. Thus, the two remaining parameters, θ and z can be used for representation of this cylinder as $x = 2\cos\theta$, $y = 2\sin\theta$, $z = z$ or, written shortly as $\mathbf{x} = (2\cos\theta, 2\sin\theta, z)$.

4. Similarly, the cylinder $y^2 + z^2 = 4$ can be parametrized by $\mathbf{x} = (x, 2\cos\theta, 2\sin\theta)$.

Spherical coordinates. If $P = (x, y, z)$ is a point in space and O denotes the origin, let

- r denotes the distance of the point $P = (x, y, z)$ from the origin O. Thus,
$$x^2 + y^2 + z^2 = r^2;$$
- θ is the angle between the projection of vector $\overrightarrow{OP} = (x, y, z)$ on the xy-plane and the vector $\overrightarrow{i} = (1, 0, 0)$ (positive x axis); and
- ϕ is the angle between the vector $\overrightarrow{OP}$ and the vector $\overrightarrow{k} = (0, 0, 1)$ (positive z-axis).



With this notation, spherical coordinates are (r, θ, ϕ). The conversion equations are

$$x = r\cos\theta\sin\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi.$$

In this parametrization, the north pole of a sphere centered at the origin corresponds to value $\phi = 0$, the equator to $\phi = \frac{\pi}{2}$ and the south pole to $\phi = \pi$. To match the geographical latitude (for which north and south pole correspond to $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ and equator to $\phi = 0$), the angle ϕ is often considered to be the angle between the equator and the vector $\overrightarrow{OP}$. In this case, $\cos\phi$ and $\sin\phi$ are switched in the equations of the spherical coordinates and we obtain

$$x = r\cos\theta\cos\phi \quad y = r\sin\theta\cos\phi \quad z = r\sin\phi.$$

The angle θ corresponds to the geographical longitude and the angle ϕ corresponds to the geographical latitude.

For example, the sphere $x^2 + y^2 + z^2 = 9$ has representation as $\mathbf{x} = (3\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi)$ or, using the second version, as $\mathbf{x} = (3\cos\theta\cos\phi, 3\sin\theta\cos\phi, 3\sin\phi)$.

The upper half of the sphere $x^2 + y^2 + z^2 = 9$ can be parametrized on several different ways.

- Using x, y as parameters $\mathbf{x} = (x, y, \sqrt{9 - x^2 - y^2})$ for $x^2 + y^2 \leq 9$.
- Using cylindrical coordinates $\mathbf{x} = (r\cos\theta, r\sin\theta, \sqrt{9 - r^2})$ for $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq 9$.
- Using spherical coordinates $\mathbf{x} = (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi)$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.

Tangent Plane

For a parametric surface

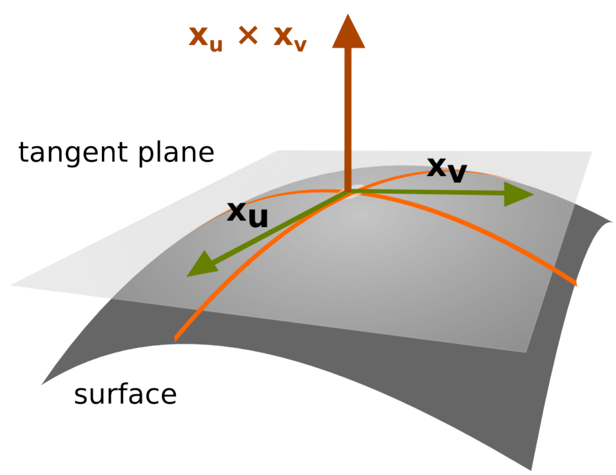
$$\mathbf{x} = (x(u, v), y(u, v), z(u, v)),$$

the derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ are vectors in the tangent plane. Thus, their cross product

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = (x_u, y_u, z_u) \times (x_v, y_v, z_v)$$

is perpendicular to the tangent plane.

If a surface is given by implicit function $F(x, y, z) = 0$, then this cross product also corresponds to the **gradient** ∇F of F.



Practice Problems.

1. Find an equation of the plane through the point $(6, 3, 2)$ and perpendicular to the vector $(-2, 1, 5)$. Check if $(2, -1, 0)$ and $(1, -2, 1)$ are in that plane.
2. Sketch the following surfaces.

   (a) $z = 6 - 2x - 3y$ (b) $z = \sqrt{9 - x^2 - y^2}$ (c) $z = \frac{1}{x^2+y^2}$ (d) $z = y^2$

3. Find the equation of the tangent plane to a given surface at the specified point.

   (a) Hyperbolic paraboloid $z = y^2 - x^2$, at $(-4, 5, 9)$
   (b) Ellipsoid $x^2 + 2y^2 + 3z^2 = 21$, at $(4, -1, 1)$.
   (c) Parametric surface given by $x = u + v\ y = 3u^2\ z = u - v$, at $(2, 3, 0)$
   (d) The cylinder $x^2 + z^2 = 4$, at $(0, 3, 2)$.

Solutions. (1) $-2(x-6) + 1(y-3) + 5(z-2) = 0 \Rightarrow -2x + y + 5z = 1$. No. Yes.
(2) (a) Plane (b) Upper hemisphere centered at the origin of radius 3.
(c) Rotate $z = \frac{1}{y^2}$ about z-axis.
(d) Cylindrical surface, translate the parabola $z = y^2$ in yz-plane along x-axis.

(3) (a) $\mathbf{x} = (x, y, y^2 - x^2) \Rightarrow \mathbf{x}_x \times \mathbf{x}_y = (2x, -2y, 1)$. Alternatively, consider $F = z - y^2 + x^2$ and find the gradient ∇F to be $(2x, -2y, 1)$. At $x = -4, y = 5$ this vector is $(-8, -10, 1)$. So the tangent plane is $-8(x+4) - 10(y-5) + z - 9 = 0 \Rightarrow -8x - 10y + z = -9$ or $8x + 10y - z = 9$.

(b) Consider $F = x^2+2y^2+3z^2-21 = 0$ and find $\nabla F = (2x, 4y, 6z)$. At $(4, -1, 1)$, $\nabla F = (8, -4, 6)$. The equation of the plane is $8(x-4)-4(y+1)+6(z-1) = 0 \Rightarrow 8x-4y+6z = 42 \Rightarrow 4x-2y+3z = 21$.

(c) $\mathbf{x} = (u + v, 3u^2, u - v) \Rightarrow \mathbf{x}_u \times \mathbf{x}_v = (-6u, 2, -6u)$. Then note that at $(2, 3, 0)$, the values of parameters are $u = 1$ and $v = 1$ so the normal vector is (-6,2,-6) The equation of the plane is $-6(x-2) + 2(y-3) - 6(z-0) = 0 \Rightarrow -6x + 2y - 6z = -6 \Rightarrow 3x - y + 3z = 3$.

(d) You can parametrize the cylinder as $\mathbf{x} = (2\cos t, y, 2\sin t)$ and calculate $\mathbf{x}_t \times \mathbf{x}_y$ to be $(-2\cos t, 0, -2\sin t)$. Note that at $(0, 3, 2)$, the values of parameters are $t = \frac{\pi}{2}$ and $y = 3$ so the normal vector is $(0, 0, -2)$. The equation of the plane is $0(x-0) + 0(y-3) - 2(z-2) = 0 \Rightarrow z = 2$.

Curvature and Theorema Egregium

For the concept of the curvature of curve $\boldsymbol{\gamma}$ to be defined, we had to ensure that we can define a unit-length tangent vector at every point. This condition was ensured by requiring that the derivative $\frac{d\boldsymbol{\gamma}}{dt} \neq 0$. Analogously, for surfaces we want to insure that the tangent plane at every point is defined (i.e. that is not collapsed into a line or a point). Since the normal vector to the tangent plane of a parametric surface $\mathbf{x}$ is given by $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}$, we want to impose a condition that guarantees that this vector is non-zero i.e.,

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}.$$

Coordinate Patches. The condition $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$ guarantees that the vectors $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ are not on the same line. Thus, they are linearly independent and they constitute a **basis of the tangent plane** and every other vector in the tangent plane can be represented as a linear combination of these two vectors. [1]

[1]**Linear Algebra background.** Consider two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that do not lie on the same line. In this case, we say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent**. Consider also the plane determined by these two vectors. For arbitrary vector $\mathbf{v}$ in the plane, we can consider the projection of $\mathbf{v}$ in direction of $\mathbf{v}_1$. This projection is a multiple of $\mathbf{v}_1$. Let a denote the multiplication factor. Consider also the projection of $\mathbf{v}$ in direction of $\mathbf{v}_2$ and let b denote the the multiplication factor. Thus,

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$$

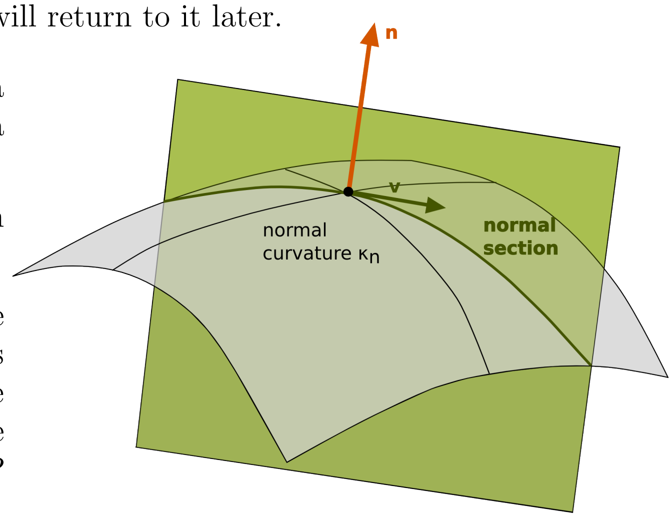
The sum $a\mathbf{v}_1 + b\mathbf{v}_2$ is called a **linear combination** of $\mathbf{v}_1$ and $\mathbf{v}_2$. This shows that every vector in the plane that we

Thus, we shall consider just surfaces such that around every point parametric equations $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ with the following properties can be found.

- The functions $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ are continuous in both variables (thus, there are no gaps or holes), one-to-one with continuous inverses (this last condition guarantees "properness").
- The partial derivatives of $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ are continuous (thus, there are no corners or sharp turns).
- The cross product $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}$ is not equal to $\mathbf{0}$ (thus, the tangent plane at each point is not collapsed into a line or a point).

We refer to those surfaces as **proper coordinate patches**. Note that it may not be possible to describe the whole surface with a single coordinate patch but it will always be possible to cover the entire surface by "patching" several different coordinate patches together. So, you can think of coordinate patches as basic surfaces that create arbitrary surface. At the moment, we will not go into the formal definition of "patching" but we will return to it later.

We present the informal idea of curvature at a point P_0 on the surface. We shall make this idea more precise during the course of the semester.



1. Take an arbitrary vector $\mathbf{v}$ of unit length in the tangent plane at P.

(2) Consider the plane determined by $\mathbf{v}$ and the normal vector of the tangent plane. This plane is perpendicular to the tangent plane and intersects the surface in a curve $\boldsymbol{\gamma}$. The curve $\boldsymbol{\gamma}$ is called the **normal section at P in the direction of $\mathbf{v}$**.

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consider can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In this case, we say that $\mathbf{v}_1$ and $\mathbf{v}_2$ **generate** the plane.

Linearly independent vectors that generate a plane are called **a basis**. Consideration of projections above demonstrates that any two linearly independent vectors in a plane constitute a basis of the plane.

For example, vectors $(1,0)$ and $(0,1)$ are a basis of xy-plane (space $\mathbb{R}^2$): these two vectors are not colinear and every vector (x,y) is the linear combination $x(1,0) + y(0,1)$.

The same concepts can be defined in three-dimensional space. any three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ that do not lie on the same plane are said to be **linearly independent**. Any other vector $\mathbf{v}$ can be expressed as a sum of its projections in directions of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$$

i.e. as a **linear combination** of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Thus, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ **generate** the space. Linearly independent vectors that generate the space are called **a basis**. By considering projections, any three linearly independent vectors in space are a basis of the space.

For example, vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are a basis of $\mathbb{R}^3$: they are not in the same plane, and every vector (x,y,z) is the linear combination $x(1,0,0) + y(0,1,0) + z(0,0,1)$.

Another example of a basis of $\mathbb{R}^3$ are the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ of the moving frame of a curve at any of its points.

(3) Compute the curvature κ of $\boldsymbol{\gamma}$ at P. The **normal curvature in the direction of v** denoted by $\kappa_n(\mathbf{v})$ is taken to be

$$\kappa_n(\mathbf{v}) = \pm\kappa.$$

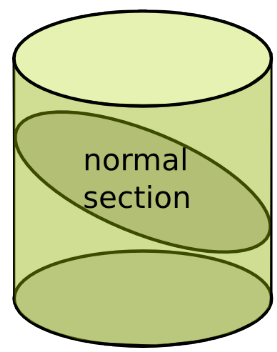
If the normal vector of the curve and the normal vector of the tangent plane have the *same direction*, then $\kappa_n(\mathbf{v}) = \kappa$ and $\kappa_n(\mathbf{v}) = -\kappa$ otherwise.

Examples. (1) Normal curvature of a **plane** in any direction is 0. This is because any plane perpendicular to the given plane intersects it in a straight line so all normal sections are lines and, thus, have zero curvature.

(2) Absolute value of the normal curvature of a **sphere** with radius a is $\frac{1}{a}$. This is because any plane perpendicular to the sphere at a point on it intersects the sphere in a great circle through the point. So, all normal sections are circles of radii a with the curvature of $\frac{1}{a}$. Thus, the normal curvature is $\pm\frac{1}{a}$.

(3) Consider the **cylinder** $x^2 + y^2 = a^2$.

At any point, a plane perpendicular to the tangent plane which is not parallel to the central axis intersects the cylinder in an ellipse. The ellipse does not have constant curvature so, in this case, the normal curvature is not constant also. In the case when we choose the plane that is parallel to the central axis, the normal section is a straight line and so the curvature in that case is zero.



We can improve this analysis if, instead of considering arbitrary normal sections, we choose to consider normal sections in two specifically chosen directions. Namely, an arbitrary vector $\mathbf{v}$ in the tangent plane can be represented as a linear combination of two specific vectors of the tangent plane: $\mathbf{v}_1$ parallel to the central axis and $\mathbf{v}_2$ perpendicular to $\mathbf{v}_1$. The normal plane corresponding to $\mathbf{v}_1$ has a straight line as the normal section and so the curvature in direction of $\mathbf{v}_1$ is 0. The normal plane corresponding to $\mathbf{v}_2$ has a circle of radius a for the normal section and so the curvature in direction of $\mathbf{v}_2$ is $\frac{1}{a}$. The absolute value of the normal curvature in direction of any other vector $\mathbf{v}$ will have a value between these two values so

$$0 \leq |\kappa_n(\mathbf{v})| \leq \frac{1}{a}.$$

This calculation can still be made more explicit by representing $\mathbf{v}$ as $\cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2$ where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}$ measured in positive direction (just as for polar coordinates when considering $\vec{i}$ and $\vec{j}$). Then it can be calculated that the normal section is an ellipse with semi-axes a and $\frac{a}{|\cos\theta|}$ and that $|\kappa_n(\mathbf{v})|$ at the relevant point is $\frac{\cos^2\theta}{a}$. Since $\cos^2\theta$ is taking values between 0 and 1, this agrees with our earlier observation that $0 \leq |\kappa_n(\mathbf{v})| \leq \frac{1}{a}$.

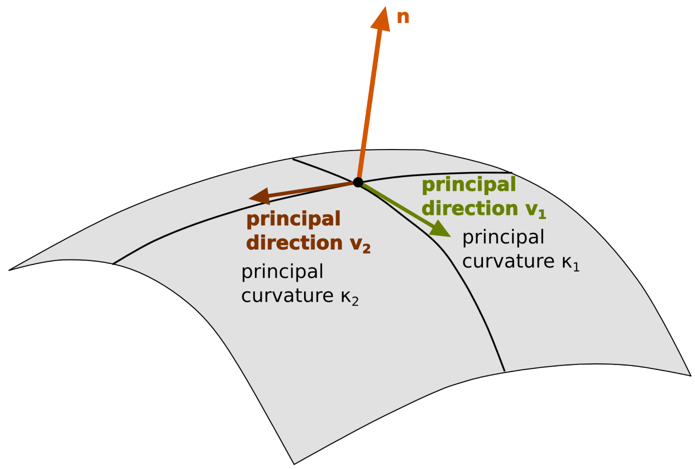
Principal curvatures. The example with cylinder is interesting because it turns to be more general than one would imagine. Namely, for every point on any surface, one can choose orthogonal unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ called **principal directions** and the normal curvatures determined by them will correspond to the **maximal and minimal values of the normal curvature** $\kappa_n(\mathbf{v})$.

These two values are denoted by κ_1 and κ_2 and are called **principal curvatures.**

The product of the principal curvatures is called

the **Gaussian curvature** $K = \kappa_1\kappa_2$.

In the example with the cylinder, the fact that at every point of the cylinder there is a direction in the tangent plane that has normal section which is a straight line is reflected in the fact that $K = 0$.

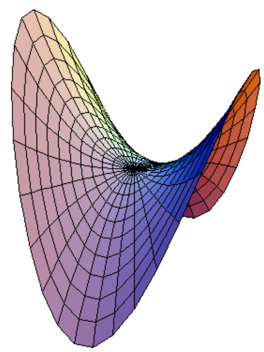


Note also that the cylinder can be slit and unrolled into a flat sheet of paper without stretching or tearing and without affecting the length any curve. A surface with this property will also have $K = 0$. This means that the geometry of the cylinder locally is indistinguishable from the geometry of a plane. In cases like this, we say that the two surfaces are **locally isometric**. Note that globally cylinder is very different from the plane, though.

Following similar reasoning, we can deduce that a cone has Gaussian curvature 0. A sphere of radius a has both the principal curvatures equal to $\frac{1}{a}$, so the Gaussian curvature is $\frac{1}{a^2}$.

Let us consider now the surface given by $z = y^2 - x^2$ called **hyperbolic paraboloid**.

This surface has a saddle point at the origin, and the principal directions are in direction of x and y axis. The normal sections in these directions are two parabolas with normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ having the opposite directions. Thus, if the normal vector of the tangent plane has the same direction as $\mathbf{N}_1$ then it has the opposite direction to $\mathbf{N}_2$ and vice versa. So, one principal curvature is positive and the other is negative resulting in a **negative Gaussian curvature** at the origin.



Hyperbolic paraboloid has $K < 0$

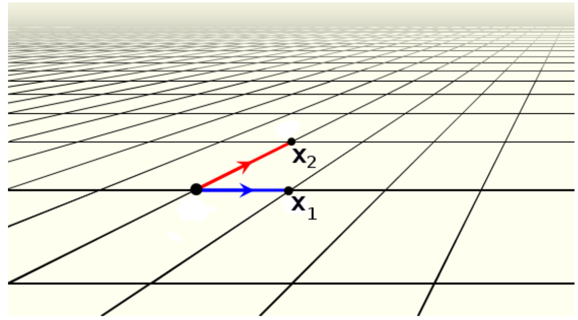
It can be shown that K is negative throughout this curve. At this point, we note a significant difference between the curvature of a curve and the normal (and Gaussian) curvature of a surface: while the curvature of a curve is defined to have just positive values, principal and Gaussian curvatures of surfaces can have negative values.

The calculation of curvature involves the surface to be embedded in the space so that normal vectors and normal sections could be considered. Thus, for "locals" on the surface, these considerations would be incomprehensible. To understand this argument, consider the fact that for people on the surface of earth, the earth appears to be flat, or the fact that in our three-dimensional world, we have hard time comprehending the four-dimensional plane perpendicular to our three-dimensional space (just as a one-dimensional curve has two-dimensional normal plane).

A concept involving only measurements on the surface (conducted by "locals") is said to be **intrinsic** while a concept whose definition involves objects external to the surface is said to be **extrinsic**. Thus, using the definition we presented, a curvature is an extrinsic concept.

If one is to generalize the concept of curvature to higher dimensions, in particular the curvature of our physical space, we would have to be able to describe curvature intrinsically.

In particular, to determine the curvature of our physical space, we do not want to rely on four-dimensional space. Also, to define the curvature of Einstein four-dimensional space-time universe, we do not want to rely on five-dimensional space.



"Locals" view surface to be flat

Fortunately for "locals", the crowning achievement of theory of surfaces states that **the curvature can be calculated intrinsically**. This means that the Gaussian curvature of a surface can be determined entirely by measuring angles, distances and their rates on the surface itself, without further reference to the particular way in which the surface is embedded into the three-dimensional space. This result, proved by Carl Friedrich Gauss, is considered to be one of foundational results in differential geometry. It is usually referred to as **Theorema Egregium** (Latin for remarkable or extraordinary theorem).

In mathematical language, the theorem also implies that the Gaussian curvature is invariant under local isometry. This means that any bending of a surface (without stretching or tearing) does not impact the Gaussian curvature. The principal curvatures do not share the property of Gaussian curvature given by Theorema Egregium – the principal curvatures do vary with bending. The fact that their product does not vary with bending makes Theorema Egregium even more remarkable.

We devote the remainder of our study of differential geometry to accomplishing the following three goals.

Goal 1 Develop apparatus that completely describes surfaces. This will be analogous to Serret-Frenet apparatus and moving frame of a curve and will lead us to first and second fundamental forms.

Goal 2 Understand the statement of Theorema Egregium in mathematical terms. This will require consideration of geodesics and the curvature tensor.

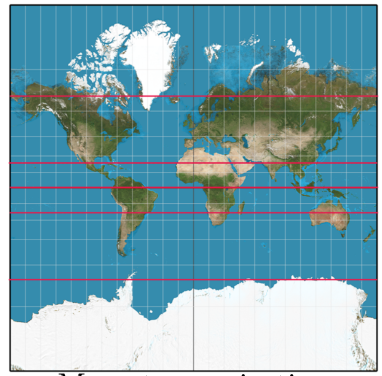
Goal 3 Theorema Egregium allows the concept of curvature to be generalized to **higher dimensions.** Two-dimensional surfaces generalize to n-dimensional **manifolds**, defined for any n and the concept of the curvature of a surfaces generalizes to the curvature of a manifold. This will enable you to understand the language used in special and general relativity. It will also enable you to generalize the content of this course to higher dimensions.

Theorema Egregium also implies that if two surfaces have different values of Gaussian curvature, than one cannot be transformed into another without tearing or crumpling. To further motivate our study, we list several corollaries of this fact.

- A sphere (with $K > 0$) and a plane (with $K = 0$) cannot be morphed one into another. Thus, a piece of paper cannot be bent onto a sphere without crumpling.

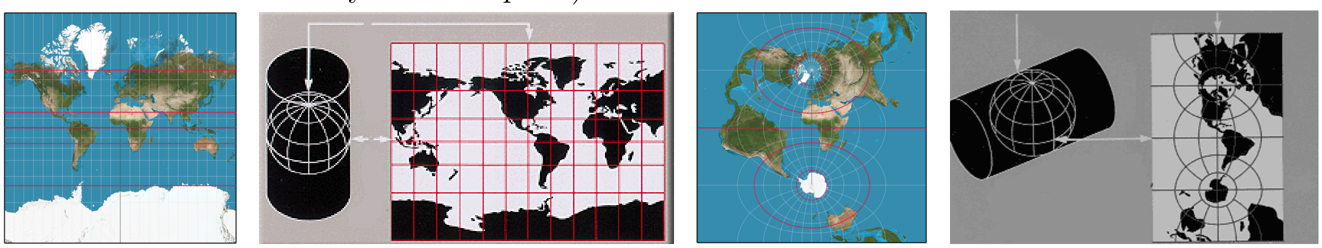
- As opposed to the cylinder (with $K = 0$), the sphere (with $K > 0$) cannot be unfolded into a flat surface. Thus, if one were to step on an empty egg shell, its edges have to split in expansion before being flattened. An orange peel can be flattened just with tearing or stretching.

As a consequence of previous observations, the Earth cannot be displayed on a map without distortion. Thus, no perfect map of Earth can be created, even for a portion of the Earth's surface and every cartographic projection necessarily distorts at least some distances. This fact is of enormous significance for cartography. Every distinct map projection distorts in a distinct way. The study of map projections is the characterization of these distortions.



Mercator projection

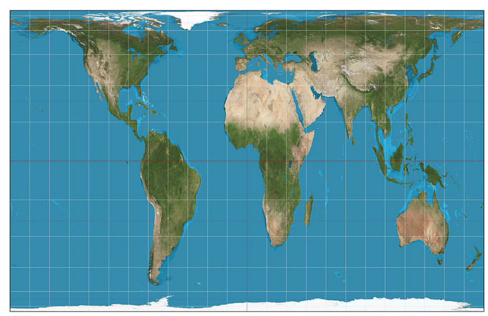
A frequently used projection, Mercator projection, preserves angles but fails to preserve area (that is why the areas around north and south pole look disproportionately large compared to the areas further away from the poles).



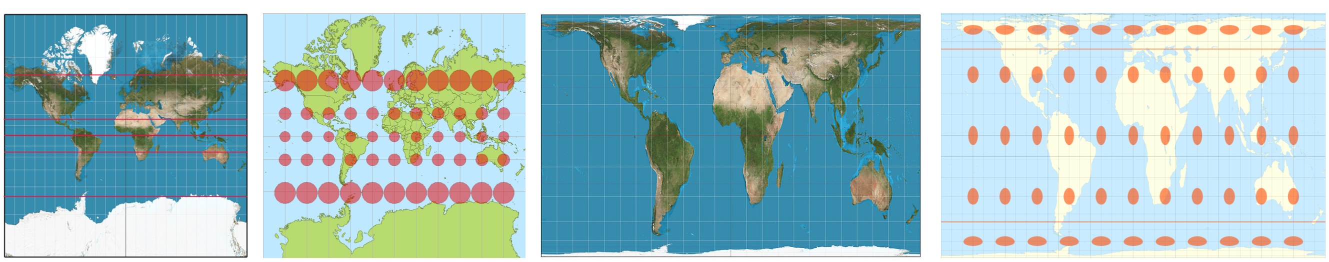
Normal and Transverse Mercator projections

The controversy surrounding the Mercator projections arose from political implications of map design since representing some countries larger than the others may implied that some are less significant.

Another projection used in some cases is Gall-Peters projection (you can see it in some world maps on airplanes). On this projection areas of equal size on the globe are also equally sized on the map. This has a consequence that areas around the equator looks elongated when compared to areas with larger geographical width.



Gall-Peters projection



Mercator and Gall-Peters with their deformations

- When trying to preserve precious toppings on a slice of pizza, you are using Theorema Egregium too: you bend a slice horizontally along a radius so that non-zero principal curvatures are created along the bend, dictating that the other principal curvature at these points must be zero. This creates rigidity in the direction perpendicular to the fold and it prevents the toppings from falling off.

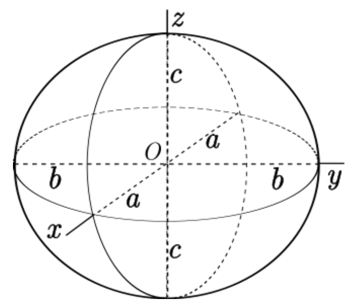
- Theorema Egregium also implies that we can measure the curvature of the Earth without leaving the surface (for example in an airplane to observe the curving) just measuring the distances and angles on the surface of the Earth.

Practice Problems.

1. The **mean curvature** is defined as the mean of the principal curvatures $H = \frac{\kappa_1+\kappa_2}{2}$. Determine the absolute value of the mean curvature of the surfaces discussed in this section: plane, sphere of radius a and cylinder $x^2+y^2=a^2$.

2. Find the Gaussian curvature of the hyperbolic paraboloid $z = y^2 - x^2$ at the origin using that the principal directions are the directions of positive x and y axis.

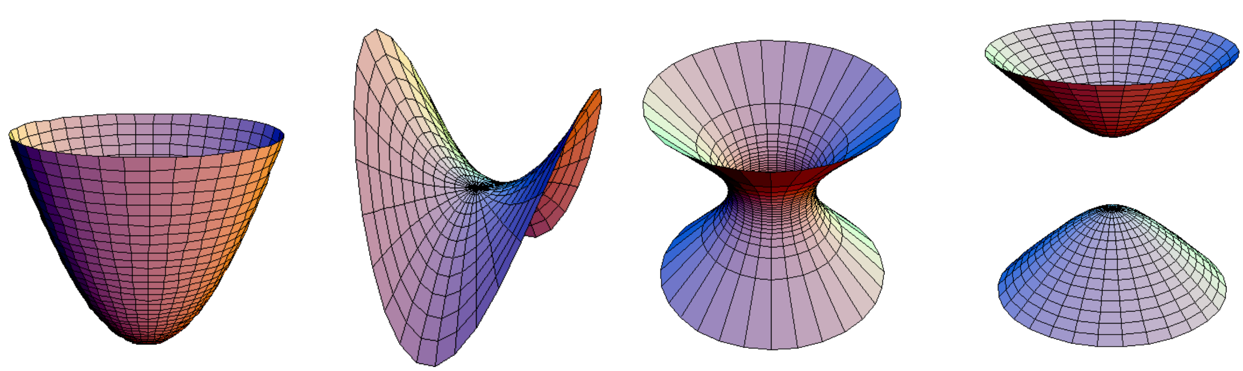
3. Find the Gaussian curvature of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the end points of the three semi-axes $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$.



Ellipsoid

4. A **quadratic surface** is any surface given by equation $ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$. By making a suitable change of variables to eliminate some terms, any quadratic surface can be put into a certain normal form. It turns out that there are 16 such normal forms. Of these 16 forms, five are non-degenerate, and the remaining are degenerate forms: cones ($\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$), cylindrical surfaces (elliptic, hyperbolic and parabolic cylinder), planes, lines, points or even no points at all. Using argument similar to those used to show that the Gaussian curvature of a cylinder is 0, deduce that K of all the degenerate quadratic surfaces is 0.

   The five non-degenerate surfaces are: ellipsoid ($\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$), elliptical paraboloid ($\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$), hyperbolic paraboloid ($\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$), hyperboloid of one sheet ($\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$) and hyperboloid of two sheets ($\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$).



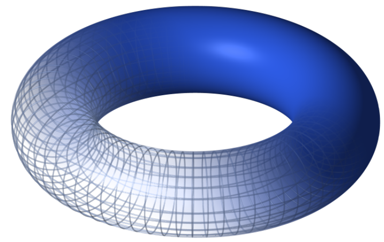
Elliptical and hyperbolic paraboloids and hyperboloids of one and two sheets

Determine the sign of Gaussian curvature for five non-degenerate quadratic surfaces.

5. A **torus** is a surface obtained by revolving one circle along the other circle creating a doughnut-like shape. Consider revolving a circle $(x-a)^2+z^2 = b^2$ in xz-plane along the circle $x^2+y^2 = a^2$ in xy-plane.

   Assume that $a > b$ so that "the doughnut" that you obtain really has a hole in the middle.

   Calculate the Gaussian curvature at any point on the "outer" circle (obtained by revolving the point $(a+b,\ 0,\ 0)$ about z-axis) and at any point on the "inner" circle (obtained by revolving the point $(a-b,\ 0,\ 0)$ about z-axis.



Torus

Solutions. (1) $H = \frac{0+0}{2} = 0$ for any plane, $|H| = \frac{\frac{1}{a}+\frac{1}{a}}{2} = \frac{1}{a}$ for the sphere of radius a, and $|H| = \frac{0+\frac{1}{a}}{2} = \frac{1}{2a}$ for the cylinder $x^2+y^2 = a^2$.

(2) In section on curves, we computed the curvature of parabola $y = x^2$ at $x = 0$ to be 2. In a similar manner we can obtain the curvature of parabola $y = -x^2$ to be 2 as well. The normal sections of $z = y^2 - x^2$ are two parabolas in xz and yz planes. In xz plane, $y = 0$ and so $z = -x^2$ and in yz plane $x = 0$ and $z = y^2$ both with curvatures 2. The two normal vectors have the opposite direction so the the two principal curvatures are 2 and -2. Thus, $\kappa_1 = 2$, $\kappa_2 = -2$ and $K = -4$.

(3) Let us calculate the curvature of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ at $(a, 0)$ and $(0, b)$ first. Here the curve can be parametrized as $\boldsymbol{\gamma} = (a\cos t, b\sin t)$. Then $\boldsymbol{\gamma}' = (-a\sin t, b\cos t, 0)$, $\boldsymbol{\gamma}'' = (-a\cos t, -b\sin t, 0)$ $\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'' = (0, 0, ab)$. Thus $|\gamma'| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$ and $|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''| = ab$ and so $\kappa = \frac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}$. At $(a, 0)$ the value of parameter t is 0 and at $(0, b)$ the value of parameter t is $\frac{\pi}{2}$. Thus $\kappa(0) = \frac{ab}{b^3} = \frac{a}{b^2}$ and $\kappa(\frac{\pi}{2}) = \frac{ab}{a^3} = \frac{b}{a^2}$.

At $(\pm a, 0, 0)$, the normal sections are in xy and xz planes. In xy plane the normal section is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ with curvature $\frac{a}{b^2}$ at $(\pm a, 0)$. In xz plane the normal section is the ellipse $\frac{x^2}{a^2}+\frac{z^2}{c^2} = 1$ with curvature $\frac{a}{c^2}$ at $(\pm a, 0)$. The normal vectors have the same directions so the two principal curvatures have the same signs. Thus the Gaussian is $K = \frac{a^2}{b^2c^2}$.

At $(0, \pm b, 0)$, the normal sections are in xy and yz planes. In xy plane the normal section is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ with curvature $\frac{b}{a^2}$ at $(0, \pm b)$. In yz plane the normal section is the ellipse $\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ with curvature $\frac{b}{c^2}$.

The normal vectors have the same directions so the two principal curvatures have the same signs. Thus the Gaussian is $K = \frac{b^2}{a^2c^2}$. On similar manner, we obtain that the Gaussian curvature at $(0, 0, \pm c)$ is $K = \frac{c^2}{a^2b^2}$.

(4) $K > 0$ for ellipsoid. For elliptical paraboloid of the form $z = \frac{x^2}{a^2}+\frac{y^2}{b^2}$ $K > 0$. $K < 0$ for hyperbolic paraboloid and for hyperboloid of one sheet, $K > 0$ for hyperboloid of two sheets.

Hyperbolical Paraboloids

(5) At point $(a+b,0,0)$, the normal sections are in xy plane and xz plane. In xy plane, the normal section is the circle of radius $a+b$ so its curvature is $\frac{1}{a+b}$. In xz plane the normal section is the circle of radius b so its curvature is $\frac{1}{b}$. The normal vectors have the same direction. Hence, the Gaussian curvature is positive and equal to $\frac{1}{b(a+b)}$.

At point $(a-b,0,0)$, the normal sections are in xy plane and xz plane as well. In xy plane, the normal section is the circle of radius $a-b$ with curvature $\frac{1}{a-b}$. In xz plane the normal section is the circle of radius b with curvature $\frac{1}{b}$. The normal vectors have the opposite direction. Hence, the Gaussian curvature is negative and equal to $\frac{-1}{b(a-b)}$.

Using this example, we can deduce that on the outer part of the torus (obtained by revolving the right half of circle $(x-a)^2+z^2=b^2$ about z-axis) the Gaussian curvature is positive and on the inner part of the torus (obtained by revolving the left half of circle $(x-a)^2+z^2=b^2$ about z-axis) the Gaussian curvature is negative. This implies the not so obvious fact that the Gaussian curvature on the "top" and "bottom" circles (obtained by revolving points $(a,0,b)$ and $(a,0,-b)$ about z-axis) is zero.