Mesh Reconstruction from Points

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Points to Meshes

- **Simplest:** convert a heightfield to a terrain

  Regular 2D grid, pixel value = height above base plane

  Same heightfield as a terrain mesh
Points to Meshes

- Image defines height value at each vertex
- Raise vertices of uniform XY grid to these heights along Z
Thought for the Day #1

What if the heights are not sampled along a grid?
Delaunay Triangulation

- Used to generate a planar base mesh whose vertices are a set of 2D points $P$

- A triangulation is **Delaunay** if no point in $P$ lies within the circumcircle of any triangle

  - It also happens to **maximize the minimum angle** of any triangle, which is why it's useful
Computing the Delaunay triangulation

- Many different algorithms
- **One approach:** start with *any* triangulation of the points, and successively flip edges if the minimum of the 6 angles increases
  - Guaranteed to converge (since minimum angle increases)
Why this works

**Lemma:** Assume the vertices of a convex quadrilateral don't all lie on the same circle. The quad can be split into two triangles in two different ways (by drawing one of the two diagonals). Then one way satisfies the circumcircle property, the other does not. The former also has the larger minimum angle.

![Diagram showing why this works](image)

- Within triangle circumcircle
- Smaller minimum angle
- No points within triangle circumcircles
- Larger minimum angle
Lawson's Flip Algorithm

• Start with *any* triangulation of $P$...

• Here's one way (the **scan triangulation method**)
  - Sort the points $p_1 \ldots p_n$ (not all collinear and $n \geq 3$) lexicographically: $(x, y) < (u, v)$ iff $x < u$ or $(x = u$ and $y < v)$
  - Let $m$ be the minimum index such that $p_1 \ldots p_m$ are not collinear
  - Triangulate $p_1 \ldots p_m$ by connecting $p_m$ to $p_1 \ldots p_{m-1}$
  - Now, for each additional point $p_i$, connect it to all points on the convex hull of $p_1 \ldots p_{i-1}$ that it “sees”

Can run in $O(n \log n)$ time
Lawson's Flip Algorithm

- Scan triangulation produces horrible triangulations
Lawson's Flip Algorithm

- Iteratively apply Delaunay flips to improve the triangulation
  - Identify two adjacent triangles
  - Flip the shared edge if the minimum of the 6 angles increases (i.e. if the circumcircle property can be restored)

![Diagram showing Delaunay flip]

- No points within triangle circumcircles
Lawson's Flip Algorithm

- The result is a Delaunay triangulation of $P$
  
  - It turns out that any Delaunay triangulation of $P$ has the same minimum angle
  
  - ... and if no four points lie on a circle, the Delaunay triangulation is unique

*(see handouts for proof)*
Delaunay and Voronoi

- The Delaunay triangulation is the **dual** of the Voronoi diagram of $P$.

Voronoi diagram – each cell consists of points nearer the cell center than to any other point in $P$.

(Superimposed) Delaunay triangulation. For each vertex of the Voronoi diagram, there is a Delaunay face. Two faces are adjacent if the corresponding vertices are connected by a Voronoi edge.

Samuel Peterson
2D surfaces embedded in 3D

- Delaunay triangulations are defined in 2D planes
- But the Delaunay idea can be extended to 2D manifold surfaces embedded in 3D

(a) Original: \( V = 6,002 \).
(b) \( V = 6,002; \varepsilon = 0.385\% \).
(c) \( V = 15,560; \varepsilon = 0.000\% \).
(d) \( V = 7,509; \varepsilon = 0.071\% \).

Original mesh
Delaunay edge flipping
Adding vertices after flipping edges, to preserve geometry
Adding vertices while flipping edges

3D volumes

• The true 3D (volumetric) analogue of a Delaunay triangulation is a Delaunay Tetrahedralization
Boundary-Conforming Delaunay

- To generate meshes for finite element analysis and similar methods, we often want to preserve the boundary while adding vertices to the interior
  - Maximize minimum angle for high quality results

Hang Si, “TetGen, a Delaunay-Based Quality Tetrahedral Mesh Generator”, 2015
Poisson Surface Reconstruction

- Triangulation is sensitive to noise, sampling pattern, omissions etc
- Can we more robustly recover the underlying surface?

Slides adapted from Kazhdan, Bolitho and Hoppe
Implicit Function Approach

- Define a function with positive values inside the model and negative values outside
Implicit Function Approach

- Define a function with positive values inside the model and negative values outside
- Extract the zero-set
Key Idea

- Reconstruct the surface of the model by solving for the indicator function of the shape

\[ \chi_M(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{if } p \notin M \end{cases} \]

In practice, we define the indicator function to be -1/2 outside the shape and 1/2 inside, so that the surface is the zero level set. We also smooth the function a little, so that the zero set is well defined.
Challenge

• How to construct the indicator function?

Oriented points

Indicator function $\chi_M$
Gradient Relationship

- There is a relationship between the normal field at the shape boundary, and the gradient of the (smoothed) indicator function.
Integration

- Represent the point normals by a vector field $V$
- Find the function $\chi$ whose gradient best approximates $V$

$$\min_{\chi} \| \nabla \chi - V \|^2$$

Squared norm of function $F$ over domain $\Omega$

$$\| F \|^2 = \int_{\Omega} \langle F(x), F(x) \rangle \, d\sigma$$

Gradient $\nabla \chi = \left( \frac{\partial \chi}{\partial x}, \frac{\partial \chi}{\partial y}, \frac{\partial \chi}{\partial z} \right)$
Integration as a Poisson Problem

- Represent the point normals by a vector field $V$
- Find the function $\chi$ whose gradient best approximates $V$

$$\min_{\chi} \| \nabla \chi - V \|^2$$

- Applying the divergence operator, we can transform this into a Poisson problem:

$$\nabla \cdot (\nabla \chi) = \nabla \cdot V \iff \Delta \chi = \nabla \cdot V$$

Divergence $\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Laplacian $\Delta \chi = \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \chi}{\partial z^2}$
Link to Linear Least Squares

• Need to solve set of equations $Ax = b$ in a least squares sense

$$\text{minimize } ||r||^2 = ||b - Ax||^2$$

• The directional derivative in direction $\delta x$ is

$$\nabla ||r||^2 \cdot \delta x = 2\delta x^T(A^Tb - A^TAx)$$

• The minimum is achieved when all directional derivatives are zero, giving the normal equations

$$A^TAx = A^Tb$$

• Thought for the Day: Compare this equation to the Poisson equation