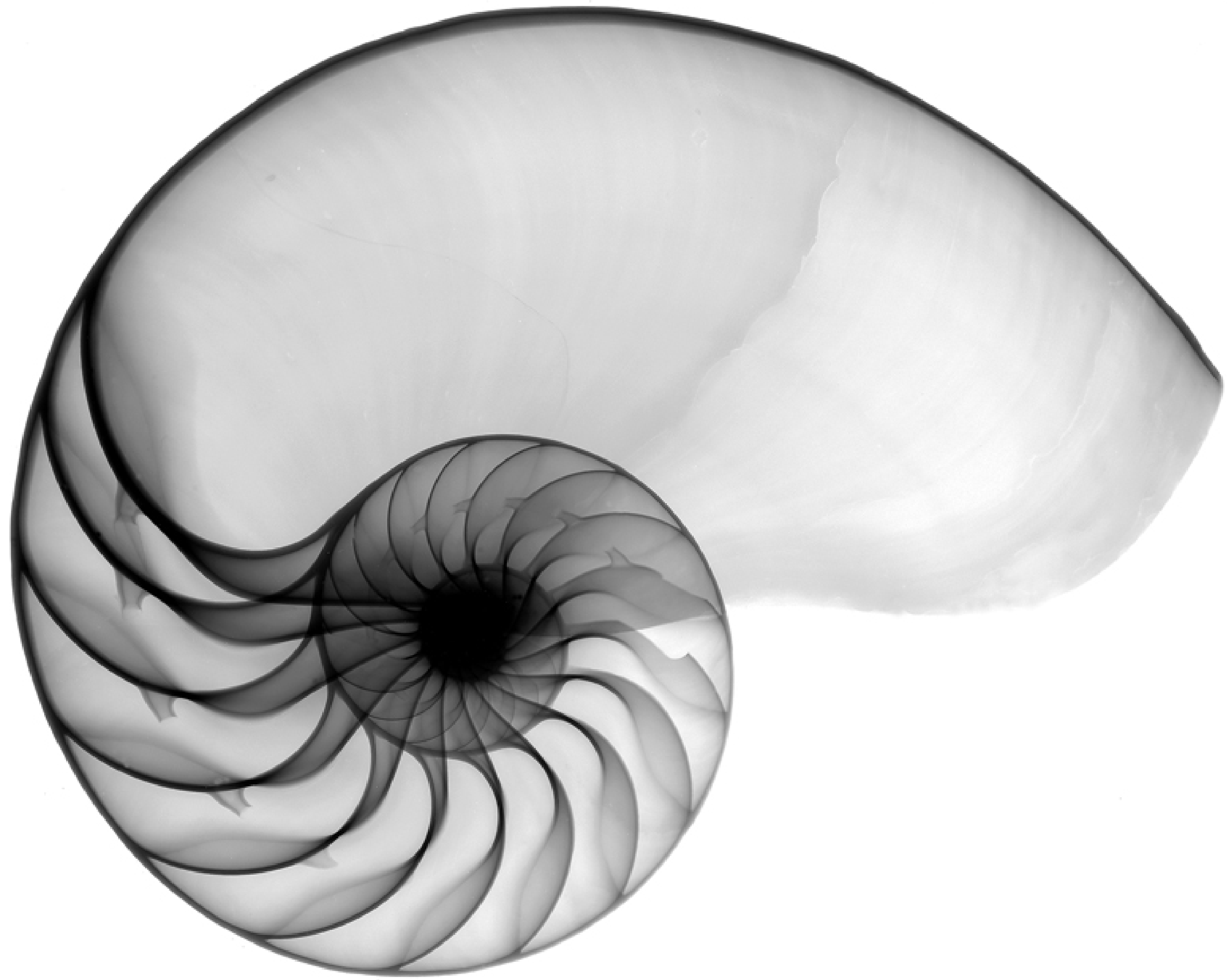
Iain Claridge



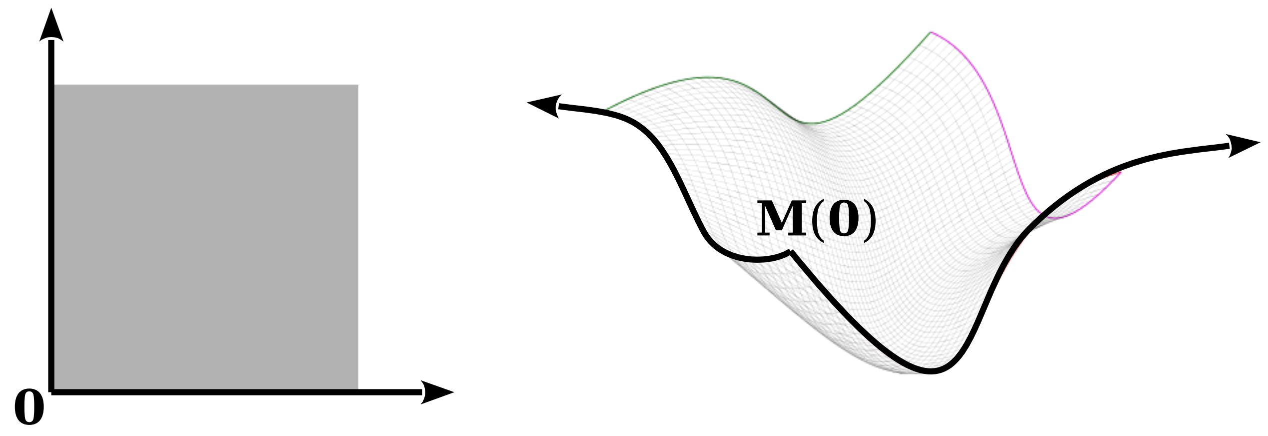
Surface Curvature

Siddhartha Chaudhuri http://www.cse.iitb.ac.in/~cs749

Curves and surfaces in 3D

- For our purposes:
  - A **curve** is a map $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ (or from some subset I of $\mathbb{R}$)

  - A **surface** is a map $\mathbf{M} : \mathbb{R}^2 \to \mathbb{R}^3$ (or from some subset Ω of $\mathbb{R}^2$)



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Curves and surfaces in 3D

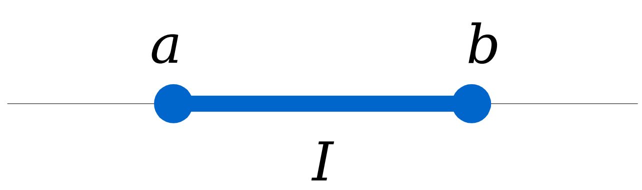
- For our purposes:
  - A **curve** is a map $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ (or from some subset I of $\mathbb{R}$)
    - $\boldsymbol{\alpha}(t) = (x, y, z)$
  - A **surface** is a map $\mathbf{M} : \mathbb{R}^2 \to \mathbb{R}^3$ (or from some subset Ω of $\mathbb{R}^2$)
    - $\mathbf{M}(u, v) = (\mathrm{x}, \mathrm{y}, \mathrm{z})$
  - We will assume everything is arbitrarily differentiable, regular, etc

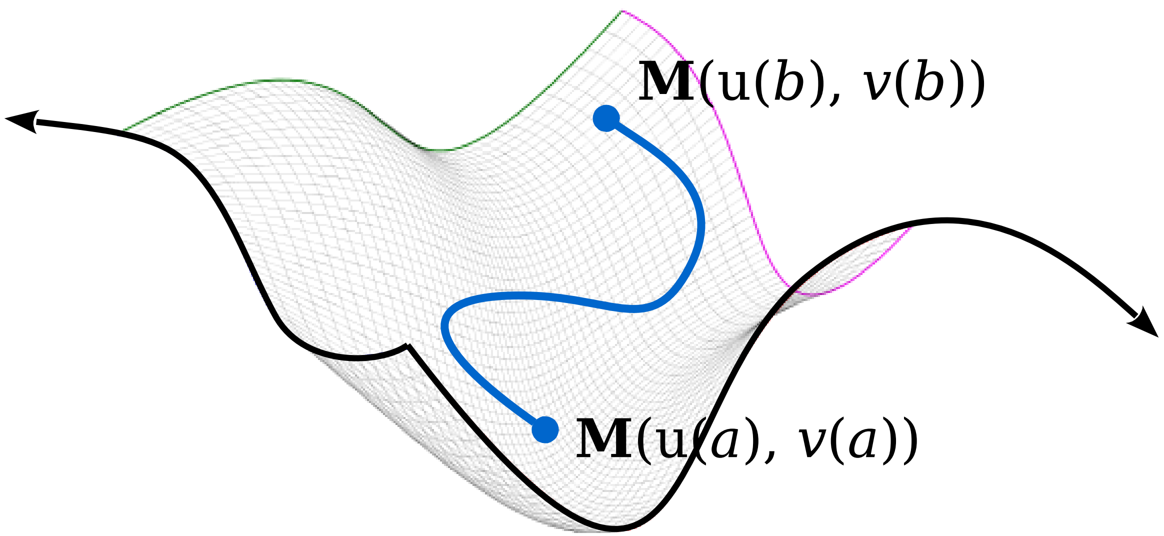
Curve on a surface

- A curve $\mathbf{C}$ on surface $\mathbf{M}$ is defined as a map

$$\mathbf{C}(t) = \mathbf{M}(u(t), v(t))$$

where u and v are smooth scalar functions





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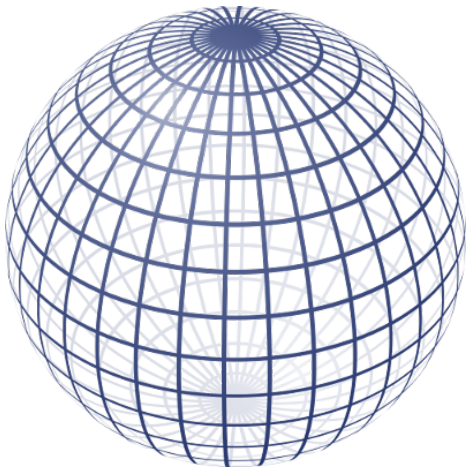
Special cases

- The curve $\mathbf{C}(v) = (u_0, v)$ for constant u_0 is called a **u-curve**
- The curve $\mathbf{C}(u) = (u, v_0)$ for constant v_0 is called a **v-curve**
- These are collectively called **coordinate curves**

- Example: coordinate curves (θ-curves and φ-curves) on a sphere



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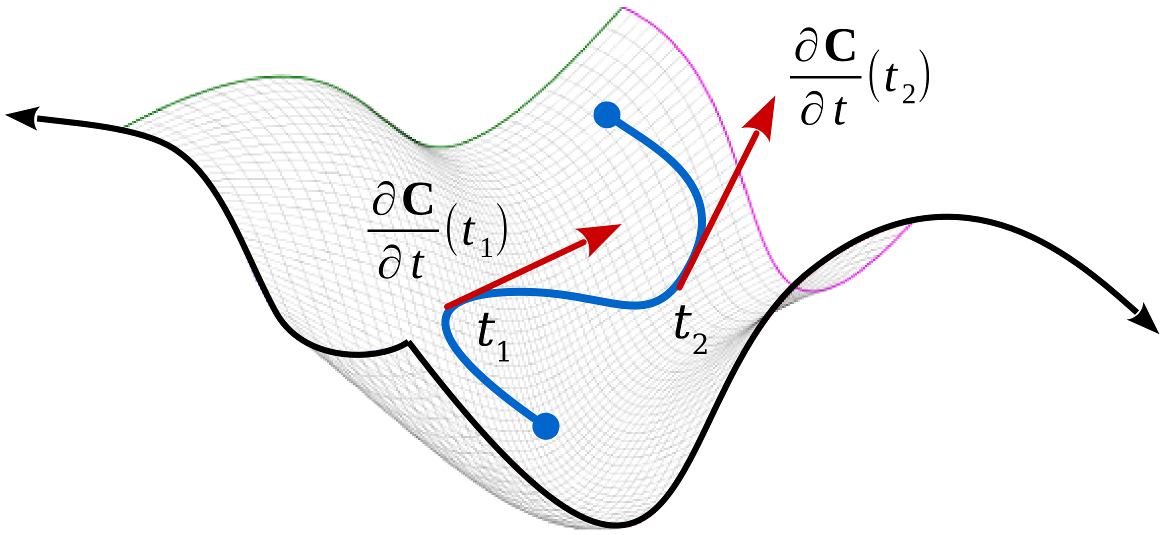
Tangent vector

- The **tangent vector** to the surface curve $\mathbf{C}$ at t can be found by the chain rule

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial \mathbf{M}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{M}}{\partial v}\frac{dv}{dt}$$



Tangent vector

- We will use the following shorthand

$$\mathbf{M}_u := \frac{\partial \mathbf{M}}{\partial u} \qquad \mathbf{M}_v := \frac{\partial \mathbf{M}}{\partial v}$$

$$\dot{u} := \frac{du}{dt} \qquad \dot{v} := \frac{dv}{dt} \qquad \dot{\mathbf{C}} := \frac{\partial \mathbf{C}}{\partial t}$$

- Then the tangent vector is $\dot{\mathbf{C}} = \mathbf{M}_u \dot{u} + \mathbf{M}_v \dot{v}$

Regular surface

- A surface $\mathbf{M}$ is **regular** if $\dot{\mathbf{C}} \neq \mathbf{0}$
  - ... for all curves $\mathbf{C} : t \mapsto \mathbf{M}(u(t), v(t))$ on the surface
  - ... such that the map $t \mapsto (u(t), v(t))$ is regular
- Eqivalently, $\mathbf{M}_u \times \mathbf{M}_v \neq \mathbf{0}$ everywhere
  - (the derivatives are not collinear)
- A point where $\mathbf{M}_u \times \mathbf{M}_v \neq \mathbf{0}$ is called a **regular point**
  - (else, it is a **singular point**)

Tangent space

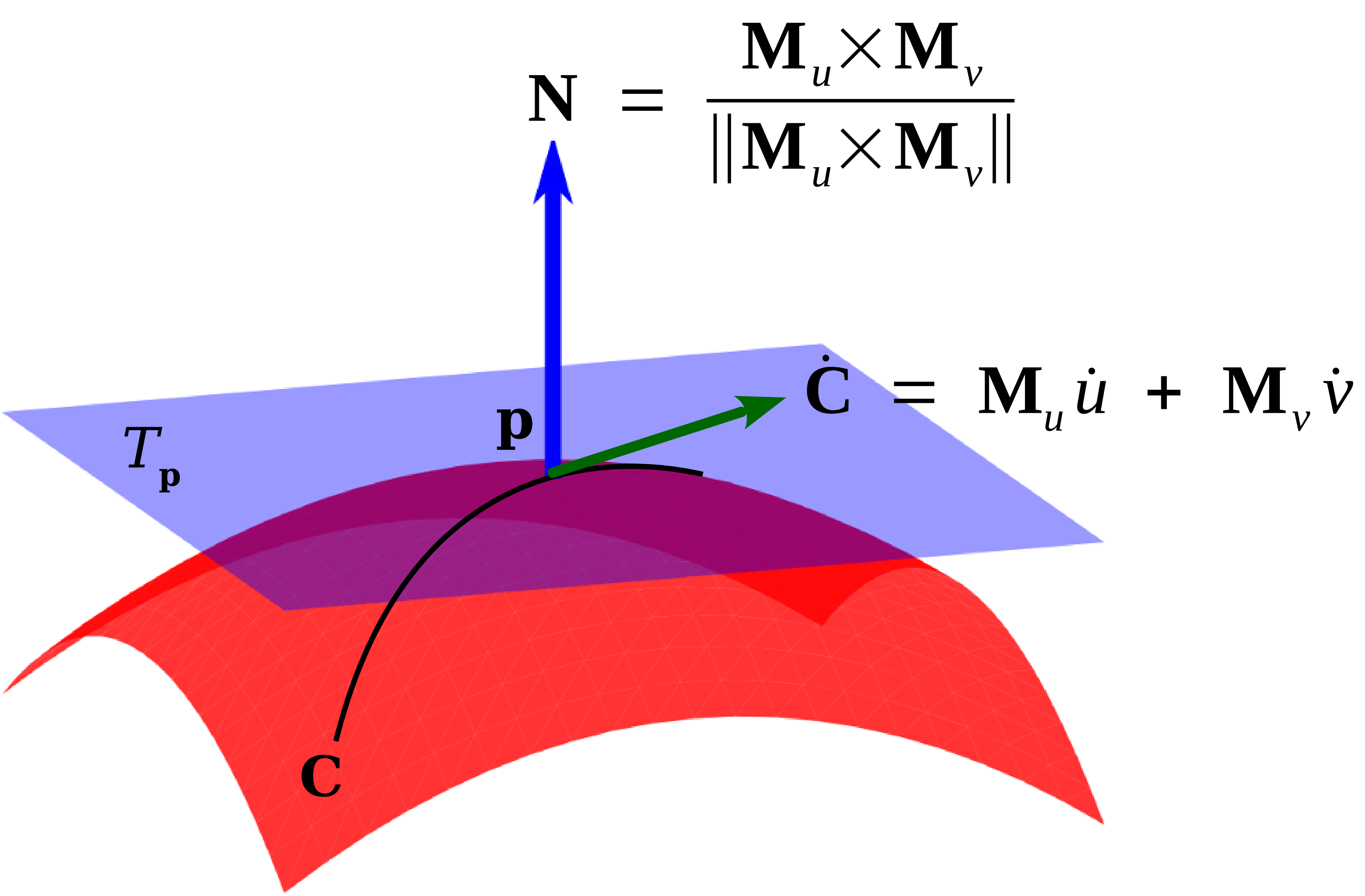
- All tangent vectors at a point $\mathbf{p}$ are of the form

$$\mathbf{M}_u \dot{u} + \mathbf{M}_v \dot{v}$$

- If the point is regular, the tangent vectors form a 2D space called the **tangent space** $T_{\mathbf{p}}$ at $\mathbf{p}$
  - $\mathbf{M}_u$ and $\mathbf{M}_v$ are basis vectors for the tangent space
- The unit normal to the tangent space, also known as the **normal** to the surface at the point, is

$$\mathbf{N} = \frac{\mathbf{M}_u \times \mathbf{M}_v}{\|\mathbf{M}_u \times \mathbf{M}_v\|}$$

Tangent space



Thought for the Day #1

If we change the parameters of the surface to, e.g.

$$u := u(r, s), \quad v := v(r, s)$$

does the normal change, and if so how?

Arc length on a surface

- Consider a curve $\mathbf{C}$ on surface $\mathbf{M}$
- Its (differential) arc length at point $\mathbf{p}$ is

$$\|\dot{\mathbf{C}}\| = \|\mathbf{M}_u \dot{u} + \mathbf{M}_v \dot{v}\|$$

- Squaring

$$\|\dot{\mathbf{C}}\|^2 = (\mathbf{M}_u \cdot \mathbf{M}_u)\dot{u}^2 + 2(\mathbf{M}_u \cdot \mathbf{M}_v)\dot{u}\dot{v} + (\mathbf{M}_v \cdot \mathbf{M}_v)\dot{v}^2$$

or

$$\|\dot{\mathbf{C}}\|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

First Fundamental Form

- The map $(x, y) \mapsto Ex^2 + 2Fxy + Gy^2$ is called the **first fundamental form** of the surface at $\mathbf{p}$

$$I_{\mathbf{p}}(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- For a regular surface, the matrix is positive definite since E (and G) > 0 and $EG - F^2 > 0$
- Because of the relation to differential arc length ds, the first fundamental form is often written as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

and called a **Riemannian metric**

Second Fundamental Form

- Consider a curve $\mathbf{C}$ on surface $\mathbf{M}$ parametrized by arc length
- Its curvature at point $\mathbf{p}$ is $\|\ddot{\mathbf{C}}\|$
- Writing $L = \mathbf{N}\cdot\mathbf{M}_{\mathrm{uu}}$, $M = \mathbf{N}\cdot\mathbf{M}_{\mathrm{uv}}$, $N = \mathbf{N}\cdot\mathbf{M}_{\mathrm{vv}}$ we have

$$\|\ddot{\mathbf{C}}\| = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

- The map $(x, y) \mapsto Lx^2 + 2Mxy + Ny^2$ is called the **second fundamental form** of the surface at $\mathbf{p}$

$$II_{\mathbf{p}}(x, y) = [x\ y]\begin{bmatrix} L & M \\ M & N \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Caution!

Remember that the fundamental forms depend on the surface point $\mathbf{p}$

The coefficients E, F, G, L, M, N are not in general constant over the surface (it would be clearer but more cluttered to write them as $E_{\mathbf{p}}, F_{\mathbf{p}}$ etc). They can take different values at different points.

Analogies with curves

Curves:

First derivative → arc length

Second derivative → curvature

Surfaces:

First fundamental form → distances

Second fundamental form → (extrinsic) curvatures

Intrinsic and Extrinsic Properties

- Properties of the surface related to the first fundamental form are called **intrinsic** properties
  - Determined only by measuring distances on the surface
- Properties of the surface related to the second fundamental form are called **extrinsic** properties
  - Determined by looking at the full embedding of the surface in $\mathbb{R}^3$

Gaussian Curvature

- The Gaussian curvature at a surface point is an intrinsic property

$$K = \frac{LN - M^2}{EG - F^2}$$

- But this involves L, M, N from the second fundamental form, how is this intrinsic?

Theorem Egregium of Gauss

- The Gaussian curvature can be expressed solely as a function of the coefficients of the first fundamental form and their derivatives

$$K = \frac{\det \begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \det \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}$$

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Intrinsic classification of surface points

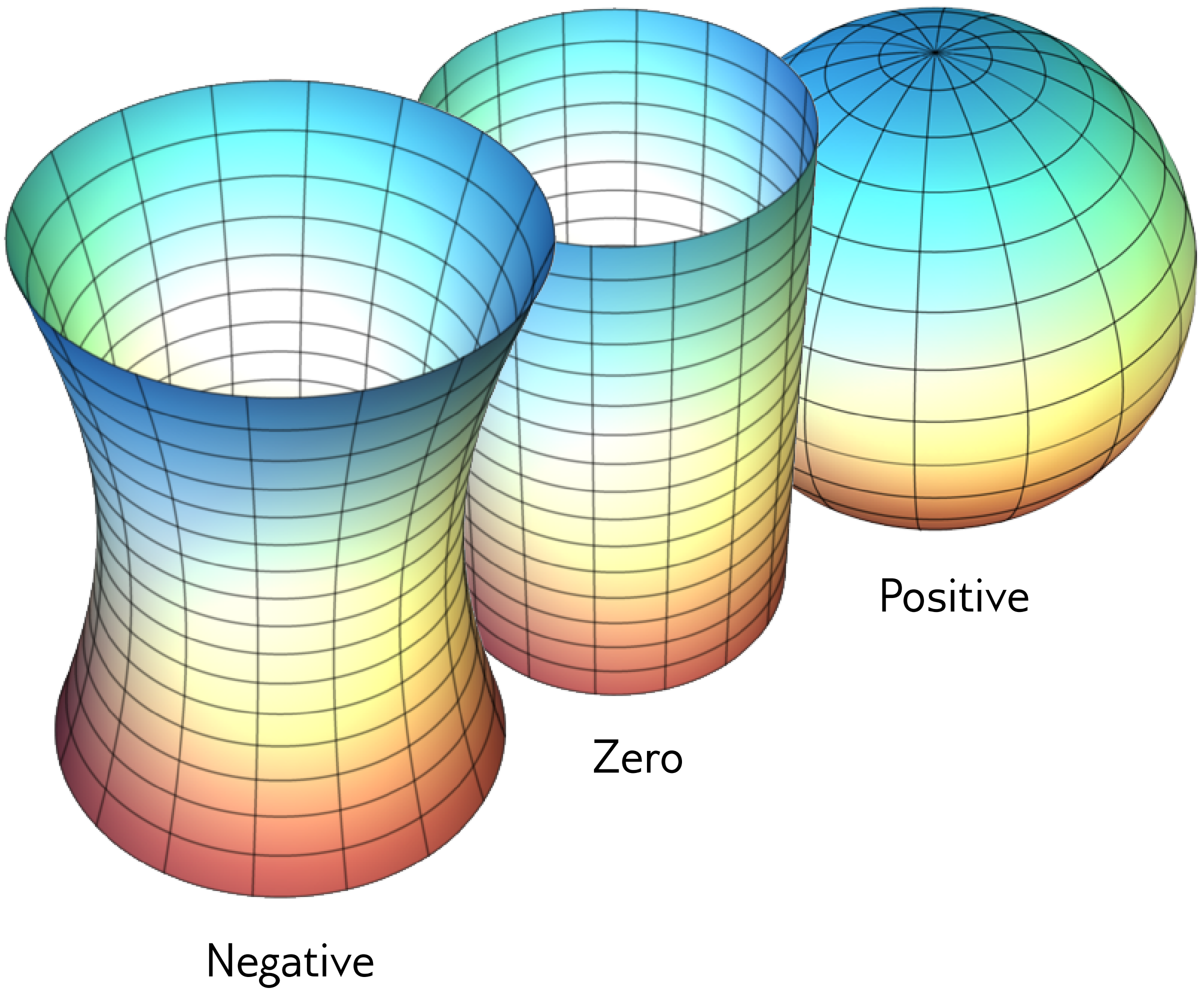
- A surface point is

(1) *Elliptic* if $LN - M^2 > 0$, or equivalently $K > 0$.

(2) *Hyperbolic* if $LN - M^2 < 0$, or equivalently $K < 0$.

(3) *Parabolic* if $LN - M^2 = 0$ and $L^2 + M^2 + N^2 > 0$, or equivalently $K = \kappa_1\kappa_2 = 0$ but either $\kappa_1 \neq 0$ or $\kappa_2 \neq 0$.

(4) *Planar* if $L = M = N = 0$, or equivalently $\kappa_1 = \kappa_2 = 0$.



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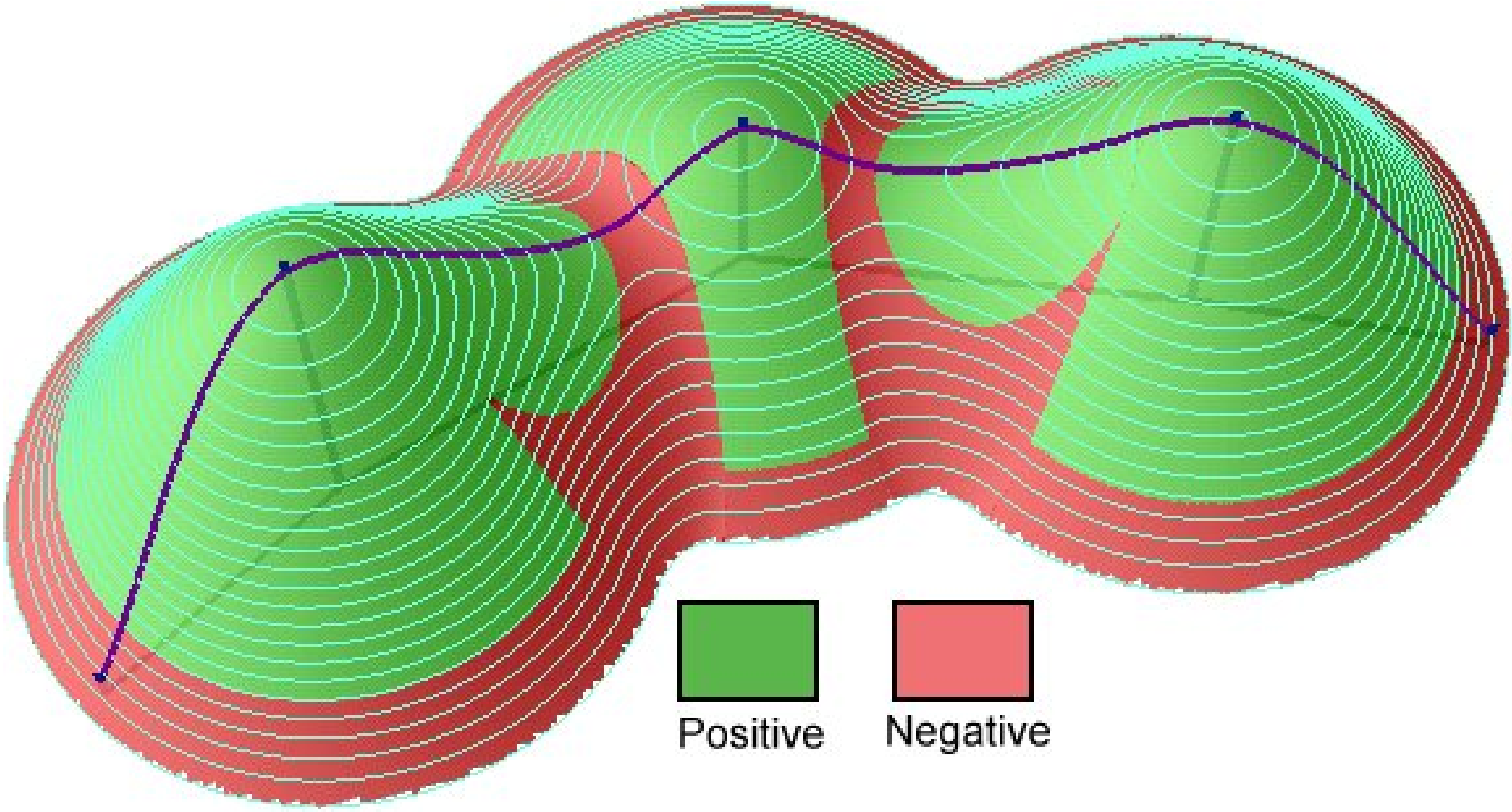
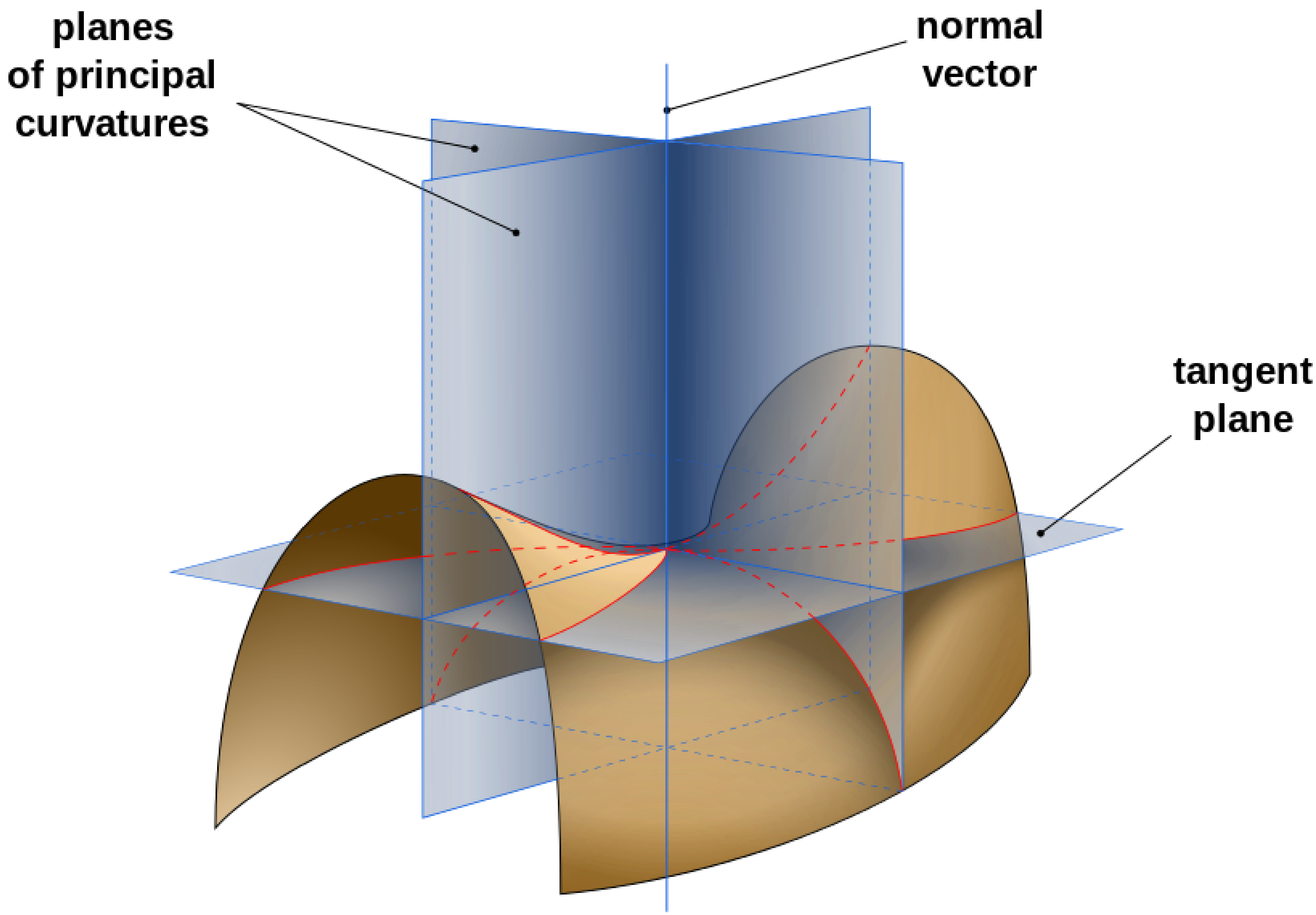

Positive
Negative

Principal Curvatures

- Geodesic curves passing through a point assume maximum and minimum curvatures in orthogonal directions
- These curvatures are called the **principal curvatures** K_1 and K_2, and the corresponding directions the **principal directions**
- The principal curvatures are **extrinsic** properties

Principal Curvatures



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Principal Curvatures

- The principal curvatures are the eigenvalues of the **shape operator**, computed from the fundamental form matrices

$$S = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{bmatrix}$$

(and the principal directions are the eigenvectors)

- It turns out that $K = K_1 K_2$

Bonnet's Theorem

A surface in 3-space is uniquely determined upto rigid motion by its first and second fundamental forms

(Compare to the Fundamental Theorem of Space Curves: curvature and torsion uniquely define a curve upto rigid motion.

An even more direct analogue is the Fundamental Theorem of Surface Theory: two fields of 2x2 matrices over a simply connected open 2D domain, that satisfy certain conditions, are the first and second fundamental form matrices of a surface uniquely defined upto rigid motion. Do you see why this theorem and Bonnet's Theorem are *not* saying the same thing?)