Sweep Surfaces for CAGD

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Outline of the work

- When introducing a new surface type in a CAD kernel
  - Parametrization: Local aspects
  - Topology: Global aspects
  - Self-intersection: Global aspects
- Parametrization: **Funnel**
- Self-intersection: **Trim curves** and **locus of** $\theta = 0$
- Topology: **Local homeomorphism** between solid and envelope.
- Further, sweeping **sharp** solids.

A simple 2-D sweep

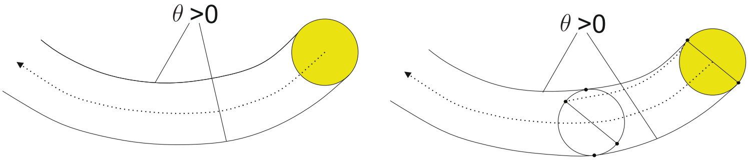


Figure: A simple 2-D sweep

A coin is translated along a parabolic trajectory in 2-D.
At each time instance t, there are *two* points-of-contact.

A non-decomposable 2-D sweep

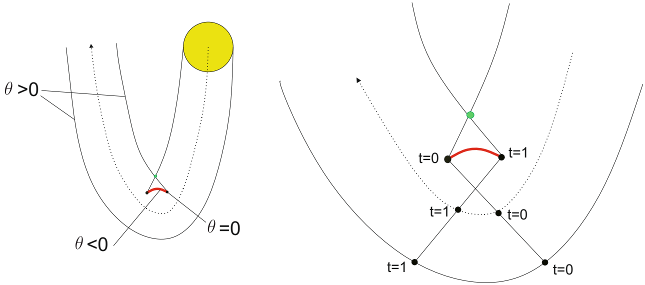


Figure: A 'non-decomposable' 2-D sweep

A coin is translated along a higher-curvature parabolic trajectory. What is the envelope in this case?

A non-decomposable 2-D sweep

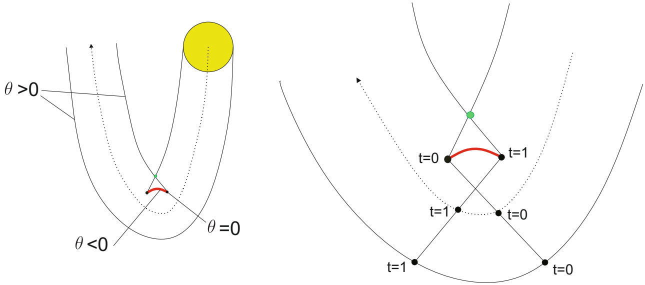


Figure: A ‘non-decomposable’ 2-D sweep

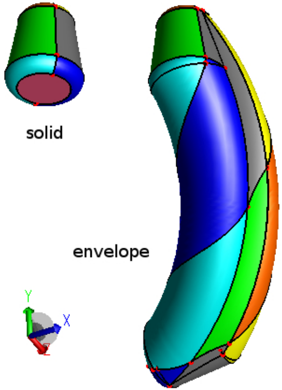
A coin is translated along a higher-curvature parabolic trajectory. What is the envelope in this case?
The parts connecting the green point to the endpoints of the red-curve also need to be trimmed to construct the correct envelope!

Envelope Definition

- Brep: A solid M in $\mathbb{R}^3$ represented by its boundary
- A trajectory in the group of rigid motions: $h : \mathbb{R} \to (SO(3), \mathbb{R}^3), h(t) = (A(t), b(t))$ where $A(t) \in SO(3), b(t) \in \mathbb{R}^3, t \in I$
- Action of h on M at time t: $M(t) = \{A(t) \cdot x + b(t) | x \in M\}$
- Trajectory of a point x: $\gamma_x : I \to \mathbb{R}^3$, $\gamma_x(t) = A(t) \cdot x + b(t)$

Envelope Definition

- **Swept volume** $\mathcal{V} := \bigcup_{t \in I} M(t)$.
- **Envelope** $\mathcal{E} := \partial \mathcal{V}$.
- **Correspondence** $R = \{(y, x, t) \in \mathcal{E} \times M \times I | y = \gamma_x(t)\}$.
- $R \subset \mathcal{E} \times \partial M \times I$.
- ∂M induces the brep structure on $\mathcal{E}$ via R.

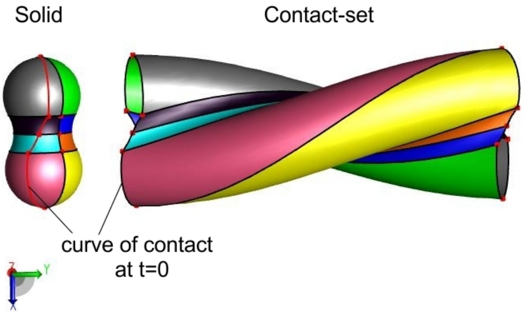


Envelope Definition

- Outward normal to ∂M at x: $N(x)$.
- Velocity of $\gamma_x(t)$: $\gamma_x'(t)$.
- Define $g : \partial M \times I \to \mathbb{R}$ as $g(x,t) = \langle A(t) \cdot N(x), \gamma_x'(t) \rangle$.
- For $I = [t_0, t_1]$, $\gamma_x(t) \in \mathcal{E}$ only if:
  (i) $g(x,t) = 0$, or
  (ii) $t = t_0$ and $g(x,t) \leq 0$, or
  (iii) $t = t_1$ and $g(x,t) \geq 0$.

Envelope Definition

- **Curve of contact** at $t_0 \in I$:
  $C(t_0) = \{\gamma_x(t_0) | x \in \partial M, g(x, t_0) = 0\}$.
- **Contact set** $C = \bigcup_{t \in I} C(t)$.



Parametrizations: Faces

- Smooth/regular surface S underlying face F of ∂M; u, v: parameters of S.
- Sweep map $\sigma : \mathbb{R}^2 \times I \to \mathbb{R}^3$
  $\sigma(u, v, t) = A(t) \cdot S(u, v) + b(t)$
- For sweep interval $I = [t_0, t_1]$, we define the following subsets of the parameter space

$$\mathcal{L} = \{(u, v, t_0) \in \mathbb{R}^2 \times \{t_0\} \text{ such that } f(u, v, t_0) \leq 0\}$$
$$\mathcal{F} = \{(u, v, t) \in \mathbb{R}^2 \times I \text{ such that } f(u, v, t) = 0\}$$
$$\mathcal{R} = \{(u, v, t_1) \in \mathbb{R}^2 \times \{t_1\} \text{ such that } f(u, v, t_1) \geq 0\}$$

- $C = \sigma(\mathcal{F})$

Parametrizations

Parametrizations

- $C = \sigma(\mathcal{F})$

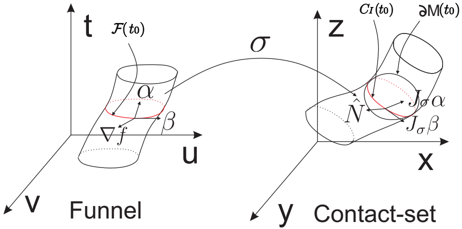
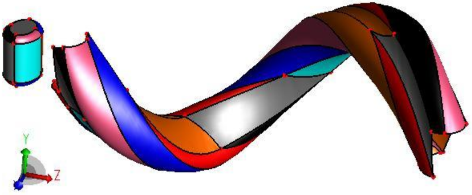


Figure: The funnel and the contact-set.

Simple sweep

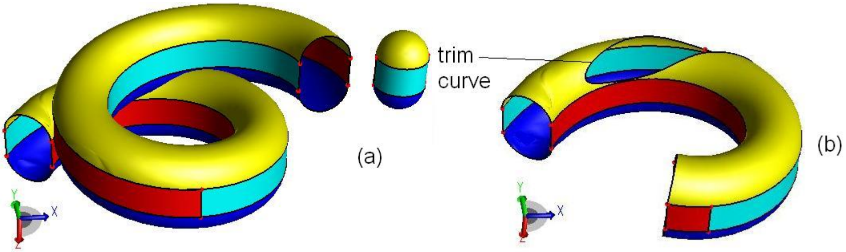
- For $t_0 \in I$, $R_{t_0} := \{(y, x, t) \in R | t = t_0\}$.
- Projections $\tau : R \to I$ and $Y : R \to \mathcal{E}$ as $\tau(y, x, t) = t$ and $Y(y, x, t) = y$.
- Sweep (M, h, I) is **simple** if for all $t \in I^o$, $C(t) = Y(R_t)$
- No trimming needed: $\mathcal{E} = \sigma(\mathcal{L} \cup \mathcal{F} \cup \mathcal{R})$.



Self-intersections

Trim set: Not all sweeps are simple

- **Trim set** $T := \{x \in C | \exists t \in I, x \in M^o(t)\}$.
- **p-trim set** $pT := \sigma^{-1}(T) \cap \mathcal{F}$.
- Clearly, $T \cap \mathcal{E} = \emptyset$.
- Extend the correspondence R to $C \times M \times I$:
  $\tilde{R} := \{(y, x, t) \in C \times M \times I | y = A(t) \cdot x + b(t)\}$.
- $\tilde{R} \not\subset C \times \partial M \times I$



Trim curves

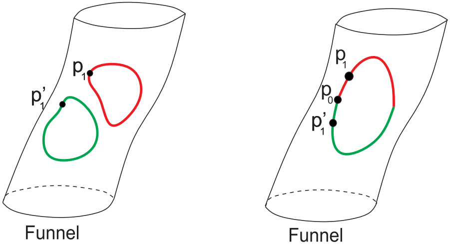
- **Trim curve** ∂T: boundary of $\overline{T}$.
- **p-trim curve**: ∂pT: boundary of $\overline{pT}$.
- For $p = (u, v, t) \in \mathcal{F}$, let $\sigma(p) = y$. $L : \mathcal{F} \to 2^{\mathbb{R}}$, $L(p) := \tau({}_y\tilde{R})$
- Define $\ell : \mathcal{F} \to \mathbb{R} \cup \infty$,

$$\begin{aligned} \ell(p) &= \inf_{t' \in L(p) \setminus \{t\}} \|t - t'\| && \text{if } L(p) \neq \{t\} \\ &= \infty && \text{if } L(p) = \{t\} \end{aligned}$$

- Define **t–sep** $= \inf_{p \in \mathcal{F}} \ell(p)$.

Trim curves

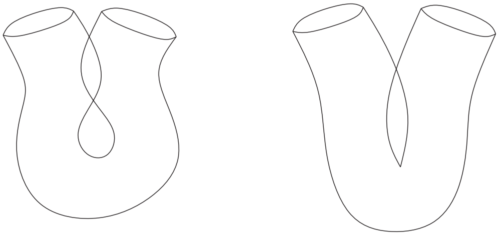
- **Elementary trim curve**: There exists $\delta > 0$ such that for all $p \in C$, $\ell(p) > \delta$.
- **Singular trim curve**: $\inf_{p \in C} \ell(p) = 0$.



(a) Elementary p-trim curves (b) Singular p-trim curves

Decomposability

- Given I, call a partition $\mathcal{P}$ of I into consecutive intervals $I_1, I_2, \ldots, I_{k_\mathcal{P}}$ to be of width δ if $\max\{length(I_1), length(I_2), \ldots, length(I_{k_\mathcal{P}})\} = \delta$.
- (M, h, I) is **decomposable** if there exists $\delta > 0$ such that for all partitions $\mathcal{P}$ of I of width δ, each sweep (M, h, I_i) is simple for $i = 1, \cdots, k_\mathcal{P}$.
- The sweep (M, h, I) is decomposable iff **t–sep** > 0. Further, if **t–sep** > 0 then all the p-trim curves are elementary.



(a) Decomposable sweep (b) Non-decomposable sweep

A geometric invariant on $\mathcal{F}$

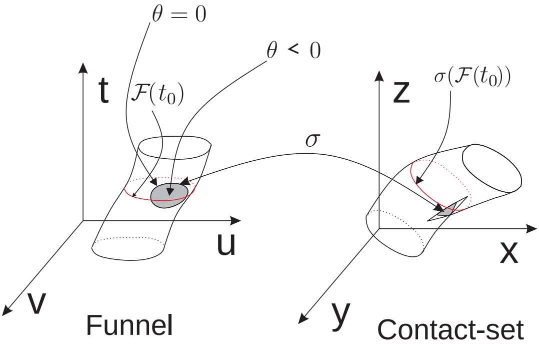
- For $p \in \mathcal{F}$, $\{\sigma_u(p), \sigma_v(p), \sigma_t(p)\}$ are l.d.
- Let $\sigma_t(p) = n(p).\sigma_u(p) + m(p).\sigma_v(p)$, n and m continuous on $\mathcal{F}$.
- Define $\theta : \mathcal{F} \to \mathbb{R}$,

$$\theta(p) = n(p) \cdot f_u(p) + m(p) \cdot f_v(p) - f_t(p)$$

- If for all $p \in \mathcal{F}$, $\theta(p) > 0$, then the sweep is decomposable. If there exists $p \in \mathcal{F}$ such that $\theta(p) < 0$, then the sweep is non-decomposable.
- θ invariant of the parametrization of ∂M.
- Arises out of relation between two 2-frames on $\mathcal{T}_C$.
- Is a non-singular function.

A geometric invariant on $\mathcal{F}$

- θ partitions the $\mathcal{F}$ into (i) $\mathcal{F}^+ := \{p \in \mathcal{F} | \theta(p) > 0\}$, (ii) $\mathcal{F}^- := \{p \in \mathcal{F} | \theta(p) < 0\}$ and (iii) $\mathcal{F}^0 := \{p \in \mathcal{F} | \theta(p) = 0\}$.
- Define $C^+ := \sigma(\mathcal{F}^+)$, $C^- := \sigma(\mathcal{F}^-)$ and $C^0 := \sigma(\mathcal{F}^0)$.



- $C^- \subset T$.
- C^0: The set of points where $dim(\mathcal{T}_C) < 2$.

Trimming non-decomposable sweeps

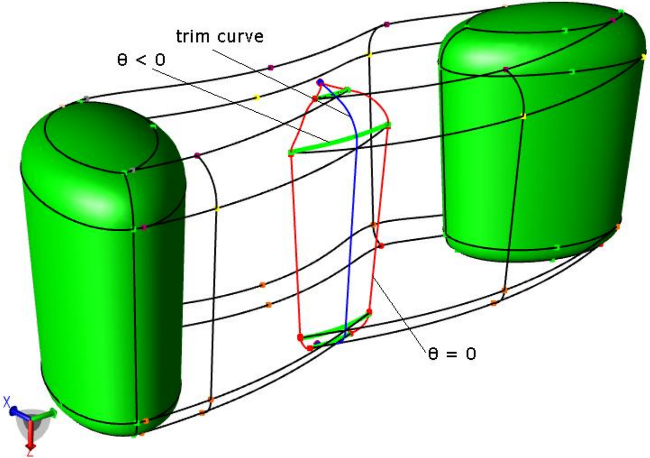
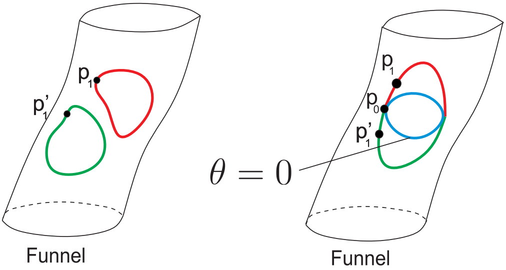


Figure: Example of a non-decomposable sweep: an elliptical cylinder being swept along y-axis while undergoing rotation about z-axis. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where θ is negative is shown in green.

Trimming non-decomposable sweeps

- If c is a singular p-trim curve and $p_0 \in c$ is a limit-point of $(p_n) \subset c$ such that $\lim_{n\to\infty} \ell(p_n) = 0$, then $\theta(p_0) = 0$.
- **singular trim point**: A limit point p of a singular p-trim curve c such that $\theta(p) = 0$.
- Every curve c of ∂pT has a curve $\mathcal{F}^0_c$ of $\mathcal{F}^0$ which makes contact with it.
- $\mathcal{F}^0$ is easy to compute since $\nabla\theta$ is non-zero.



(a) Decomposable sweep (b) Non-decomposable sweep

Locating $\mathcal{F}^0 \cap \partial pT$

- Let Ω be a parametrization of a curve $\mathcal{F}_i^0$ of $\mathcal{F}^0$. Let $\Omega(s_0) = p_0 \in \mathcal{F}_i^0$ and $\bar{z} := (n, m, -1) \in null(J_\sigma)$ at p_0, i.e., $n\sigma_u + m\sigma_v = \sigma_t$. Define the function $\varrho : \mathcal{F}^0 \to \mathbb{R}$ as follows.

$$\varrho(s_0) = \left\langle \bar{z} \times \frac{d\Omega}{ds}|_{s_0}, \nabla f|_{p_0} \right\rangle$$

- ϱ is a measure of the oriented angle between the tangent at p_0 to $\mathcal{F}_i^0$ and the kernel (line) of the Jacobian J_σ restricted to the tangent space $\mathcal{T}_{\mathcal{F}}(p_0)$.
- If p_0 is a singular trim point, then $\varrho(p_0) = 0$.

Examples of non-decomposable sweeps

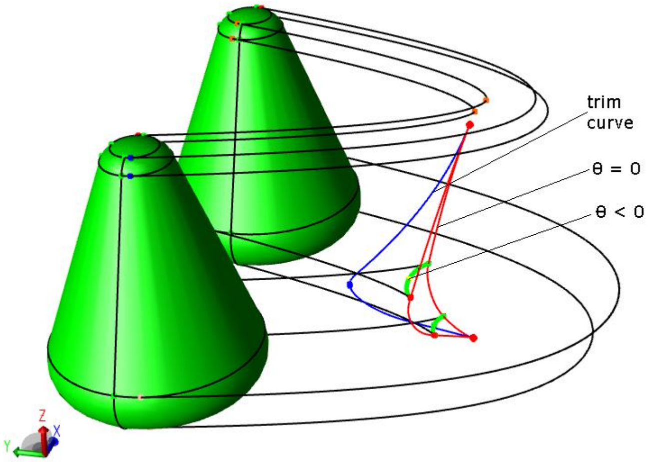


Figure: Example of a non-decomposable sweep: a cone being swept along a parabola. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where θ is negative is shown in green.

Examples of non-decomposable sweeps

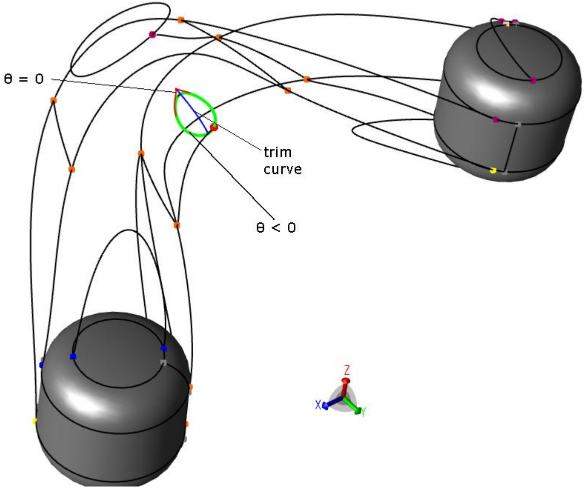


Figure: Example of a non-decomposable sweep: a cylinder being swept along a cosine curve in xy-plane while undergoing rotation about x-axis. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where θ is negative is shown in green.

Examples of non-decomposable sweeps

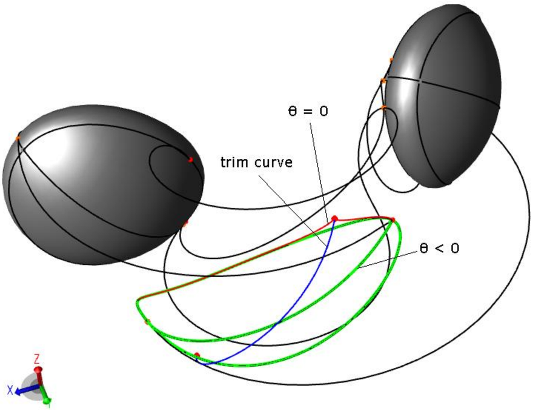


Figure: Example of a non-decomposable sweep: a blended intersection of a sphere and an ellipsoid being swept along a circular arc in *xy*-plane while undergoing rotation about *z*-axis. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where θ is negative is shown in green.

Nested trim curves

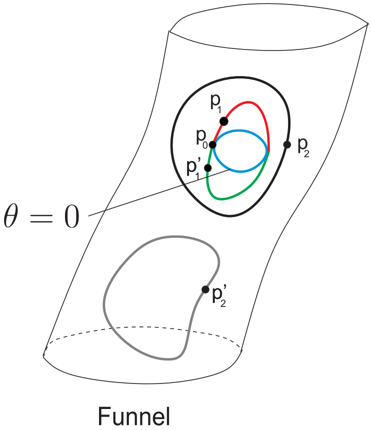


Figure: A singular p-trim curve nested inside an elementary p-trim curve

Topology

Computing topological information

- Assume w.l.o.g. (M, h, I) is simple.
- Let F be a face of ∂M and C^F be its contact set.
- The correspondence R induces the natural map $\pi : C^F \to F$ $\pi(y) = x$ such that $(y, x, t) \in R$.
- π is a well defined map.
- For $p \in \mathcal{F}^F$, let $\sigma(p) = y$. π is a local homeomorphism at y if $f_t(p) \neq 0$.
  Proof. π' is a local homeomorphism.

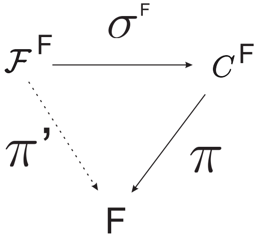
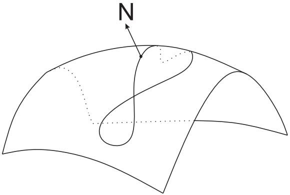
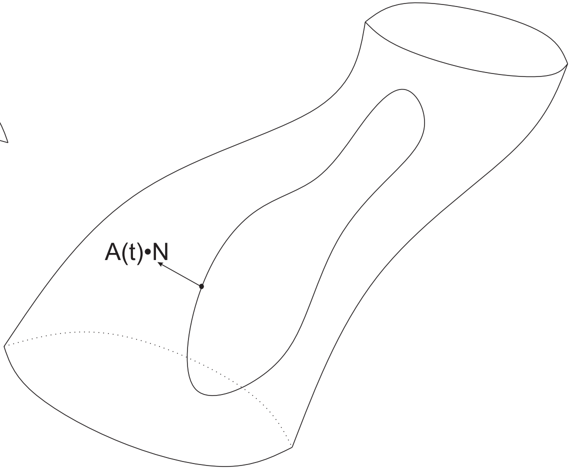


Figure: The above diagram commutes.

Orientability of the envelope



(a) ∂M



(b) Contact set

When is π orientation preserving/reversing?

- For $p \in \mathcal{F}$ let $\sigma(p) = y$ and suppose $f_t(p) \neq 0$.
- π is orientation preserving/reversing at y if $-\frac{\theta(p)}{f_t(p)}$ is positive/negative respectively.
- $-\frac{\theta}{f_t}$ is a geometric invariant.

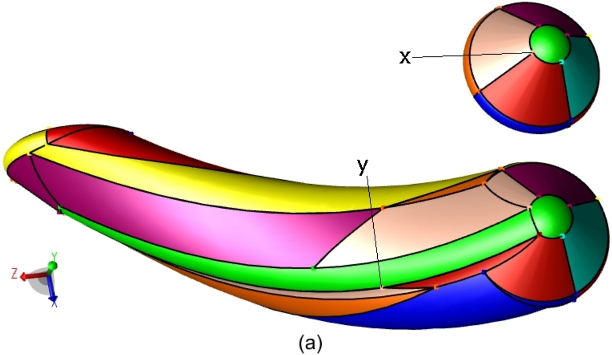


Figure: In the above example, $\pi(y) = x$. The map π is orientation preserving at y.

When is π orientation preserving/reversing?

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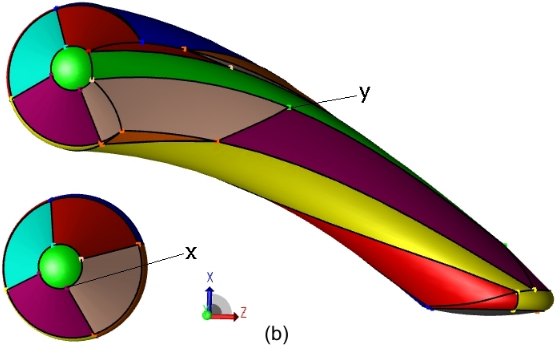


Figure: In the above example, $\pi(y) = x$. The map π is orientation reversing at y.

Geometric meaning of $-\frac{\theta}{f_t}$

- Define the following subsets of a nbhd. $\mathcal{M} \subset F(t_0)$ of a point $y \in C(t_0)$

$$f^+ = \{q \in \mathcal{M} | f(\sigma^{-1}(q)) > 0\}$$
$$f^0 = \{q \in \mathcal{M} | f(\sigma^{-1}(q)) = 0\} = C(t_0) \cap \mathcal{M}$$
$$f^- = \{q \in \mathcal{M} | f(\sigma^{-1}(q)) < 0\}$$

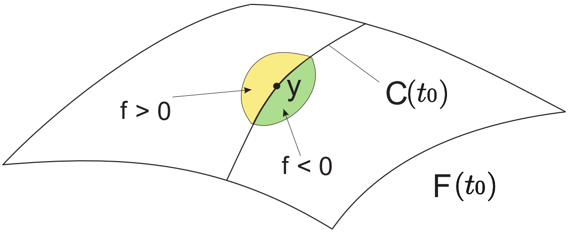


Figure: Positive and negative hemispheres at a point $y \in \partial M(t_0)$.

Geometric meaning of $-\frac{\theta}{f_t}$

- **Contributing curve** at t_0 for t is defined as the set $\{\gamma_x(t_0) | x \in \partial M, g(x,t) = 0\}$ and denoted by ${}^{t_0}C(t)$.
- ${}^{t_0}C(t_0) = C(t_0)$

Geometric meaning of $-\frac{\theta}{f_t}$

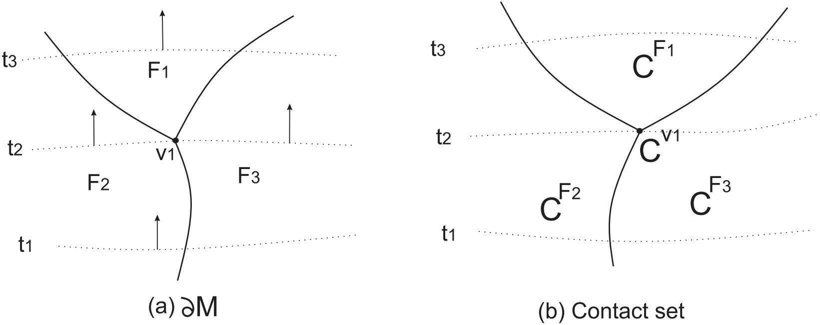


Figure: The map π is orientation preserving (a) The curves ${}^{t_0}C(t)$ are plotted on $\partial M(t_0)$ at time instances $t_1 < t_2 < t_3$. The vector $J_\sigma \cdot \alpha$ is plotted at few points. (b) The curves $C(t)$ are plotted on C at time instances $t_1 < t_2 < t_3$.

Geometric meaning of $-\frac{\theta}{f_t}$

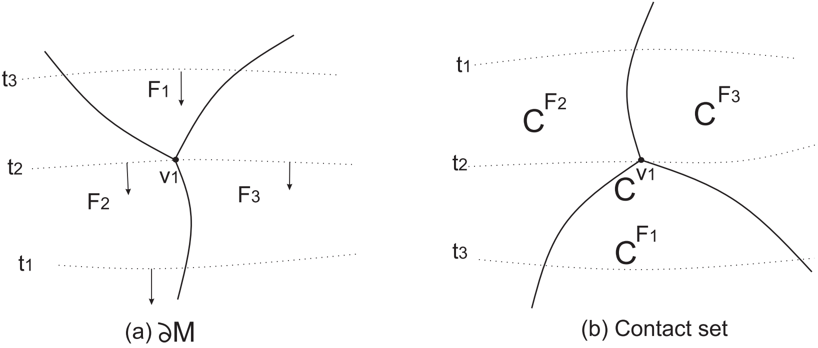


Figure: The map π is orientation reversing (a) The curves ${}^{t_0}C(t)$ are plotted on $\partial M(t_0)$ at time instances $t_1 < t_2 < t_3$. The vector $J_\sigma \cdot \alpha$ is plotted at few points. (b) The curves $C(t)$ are plotted on C at time instances $t_1 < t_2 < t_3$.

Geometric meaning of $-\frac{\theta}{f_t}$

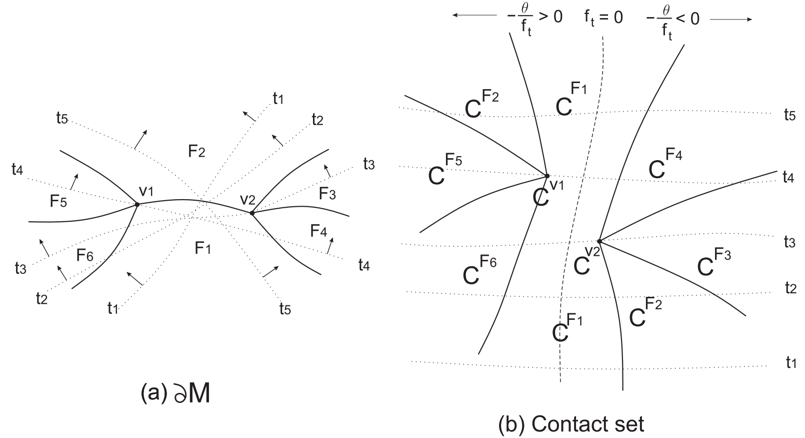


Figure: The map π is orientation preserving in a neighborhood of the point C^{v1} and reversing in a neighborhood of the point C^{v2}. (a) The curves ${}^{t_0}C(t)$ are plotted on $\partial M(t_0)$ at time instances $t_1 < t_2 < t_3 < t_4 < t_5$. The vector $J_\sigma \cdot \alpha$ is plotted at few points. (b) The curves $C(t)$ are plotted on C at time instances $t_1 < t_2 < t_3 < t_4 < t_5$.

Orienting edges of $\mathcal{E}$

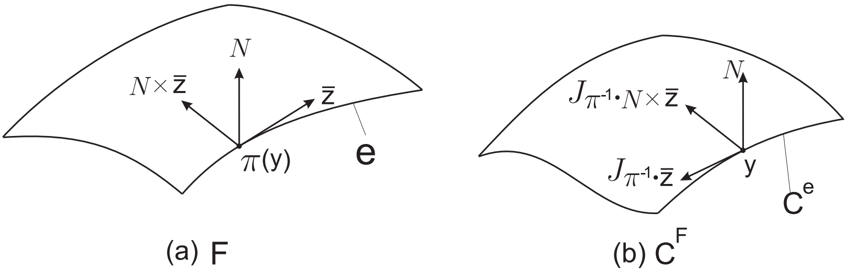


Figure: Orienting C^e. In this case $-\frac{\theta^F}{f_t^F}$ is negative at the point y.

Orienting edges of $\mathcal{E}$

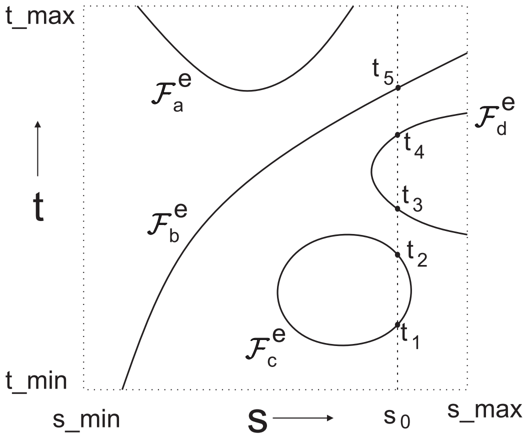


Figure: Edges in parameter space (s, t), generated by an edge $e \in \partial M$.

Computing adjacencies

- If faces C^F and $C^{F'}$ are adjacent in C then the faces F and F' are adjacent in ∂M.
- If edges C^e and $C^{e'}$ are adjacent in C then e and e' are adjacent in ∂M.
- If an edge C^e bounds a face C^F in C then the edge e bounds the face F in ∂M.
- If a vertex C^z bounds an edge C^e in C then the vertex z bounds the edge e in ∂M.
- The unit outward normal varies continuously across adjacent geometric entities in C.

Simple sweep examples

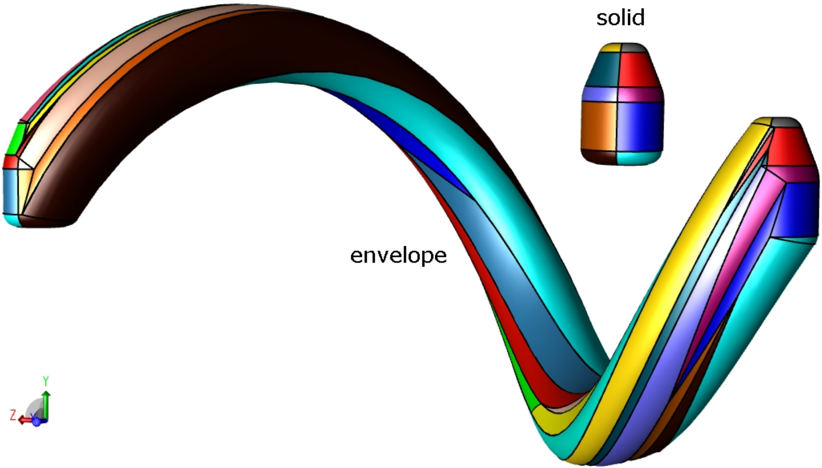
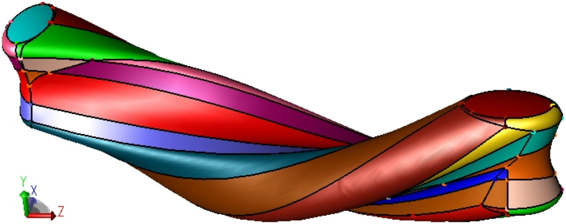
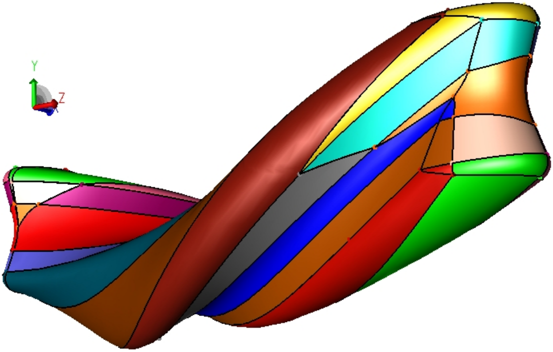


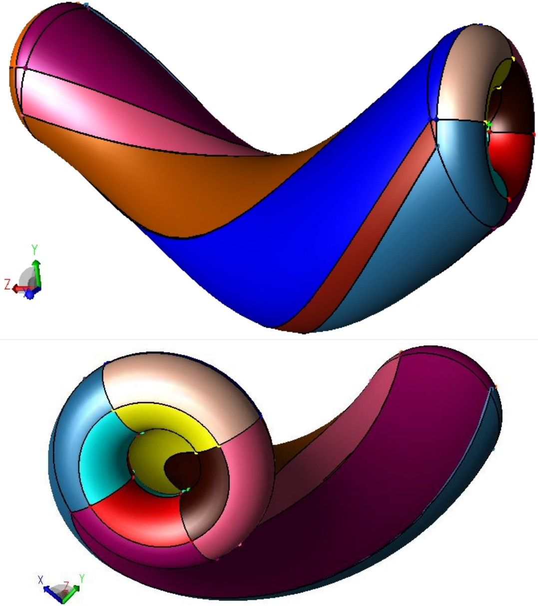
Figure: A simple bottle being swept along a screw motion with compounded rotation. Correspondence between faces of ∂M and those of the envelope is shown by color coding.

Simple sweep examples





Simple sweep examples



Overall computational framework

Algorithm 1 Solid sweep

```
for all F in ∂M do
    for all e in ∂F do
        for all z in ∂e do
            Compute vertices C^z generated by z
        end for
        Compute edges C^e generated by e
        Orient edges C^e
    end for
    Compute C^F(t_0) and C^F(t_1)
    Compute loops bounding faces C^F generated by F
    Compute faces C^F generated by F
    Orient faces C^F
end for
for all F_i, F_j adjacent in ∂M do
    Compute adjacencies between faces in C^{F_i} and C^{F_j}
end for
```

How topology of $C(t)$ varies

- $t : \mathcal{F} \to \mathbb{R}, (u, v, t) \mapsto t$ is a Morse function.
- Critical points of this function.

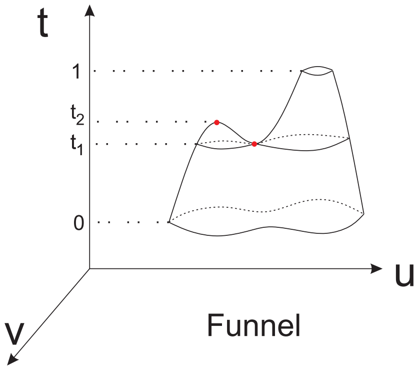
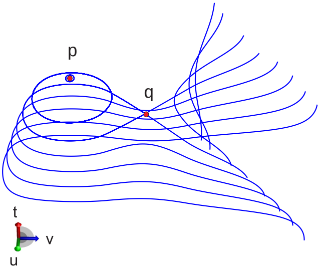
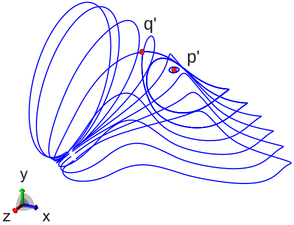


Figure: Number of connected components of $C(t)$ is $1, 2$ and 1 for $t \in (0, t_1), (t_1, t_2)$ and $(t_2, 1)$ respectively.

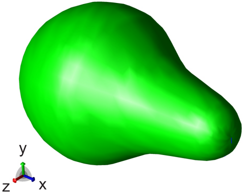
How topology of $C(t)$ varies



(a) pcurves of contact



(b) curves of contact



(c) solid

Figure: Number of connected components of $C(t)$ varies from 1 to 2 to 1 with time.

Sweeping *sharp* solids

Sweeping **sharp** solids

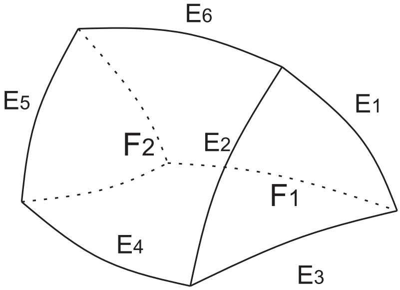
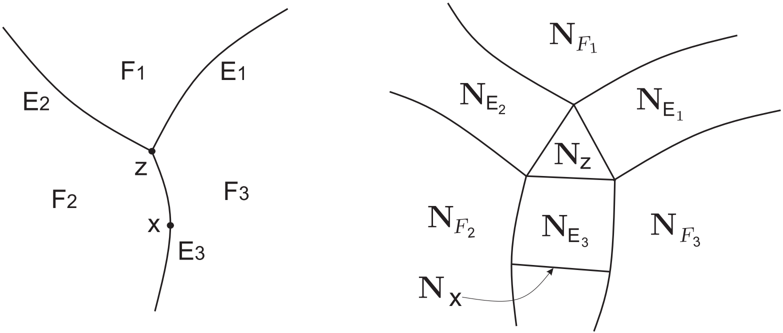


Figure: A G1-discontinuous solid.

Cone of normals and Cone bundle

- For a point $x \in \bigcap_{i=1}^{n} F_i$, define the **cone of normals** at x as $\mathcal{N}_x = \left\{ \sum_{i=1}^{n} \alpha_i \cdot N_i(x) \right\}$, where, $N_i(x)$ is the unit outward normal to face F_i at point x and $\alpha_i \in \mathbb{R}, \alpha_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$.
- For a subset X of ∂M, the **cone bundle** is defined as the disjoint union of the cones of normals at each point in X and denoted by $\mathbf{N_X}$, i.e., $\mathbf{N_X} = \bigsqcup_{x \in X} \mathcal{N}_x = \bigcup_{x \in X} \{(x, N(x)) | N(x) \in \mathcal{N}_x\}$.

Cone of normals and Cone bundle



(a) ∂M (b) $\mathbf{N}_{\partial M}$

Figure: A solid and its cone bundle.

Necessary condition

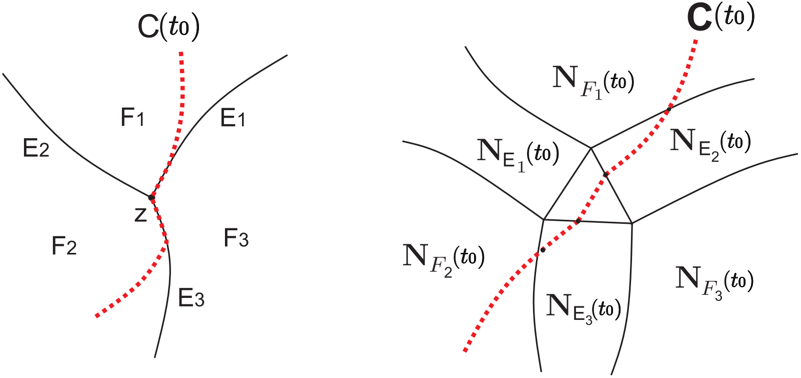
- For $(x, N(x)) \in \mathbf{N}_{\partial \mathbf{M}}$ and $t \in I$, define the function $g : \mathbf{N}_{\partial \mathbf{M}} \times I \to \mathbb{R}$ as

$$g(x, N(x), t) = \langle A(t) \cdot N(x), v_x(t) \rangle$$

- For $(y, x, t) \in R$ and $I = [t_0, t_1]$, either
(i) $t = t_0$ and there exists $N(x) \in \mathcal{N}_x$ such that $g(x, N(x), t) \leq 0$ or
(ii) $t = t_1$ and there exists $N(x) \in \mathcal{N}_x$ such that $g(x, N(x), t) \geq 0$ or
(iii) There exists $N(x) \in \mathcal{N}_x$ such that $g(x, N(x), t) = 0$.
- Projection $\pi_M : \mathbf{N}_{\partial \mathbf{M}} \to \partial M$ as $\pi_M(x, N(x)) = x$.

Necessary condition

- **Normals of contact** at t_0
  $\mathbf{C}(t_0) := \{(\gamma_x(t_0), A(t_0)\cdot N(x)) \in \mathbf{N}_{\partial\mathbf{M}}(t_0) | g(x, N(x), t_0) = 0\}$.
- **Curve of contact** at t_0 $C(t_0) := \pi_M(\mathbf{C}(t_0))$.



(a) $\partial M(t_0)$ (b) $\mathbf{N}_{\partial M}(t_0)$

Parametrization

- For x in edge $E = F_1 \cap F_2$, parametrize $\mathcal{N}_x$ with $\alpha \in [0,1]$ as $\mathcal{N}_x(\alpha) = \alpha \cdot N_1(x) + (1-\alpha) \cdot N_2(x)$
- Let I' be the domain of curve e underlying edge E.
- Define function f on the parameter space $I' \times I_1 \times I$ to $\mathbb{R}$ as $f(s,\alpha,t) = g(e(s), \mathcal{N}_{e(s)}(\alpha), t)$.
- **Funnel** $\mathcal{F} = \{(s,\alpha,t) \in I' \times I_1 \times I$ such that $f(s,\alpha,t) = 0\}$
- **Sweep map** $\sigma^e : I' \times I_1 \times I \to \mathbb{R}^6$ is defined as $\sigma^e(s,\alpha,t) = (\gamma_{e(s)}(t), A(t) \cdot \mathcal{N}_{e(s)}(\alpha))$
- **Projection** $\pi_{st} : I' \times I_1 \times I \to I' \times I$, $\pi_{st}(s,\alpha,t) = (s,t)$.
- **Projected sweep map** $\hat{\sigma}^e : I' \times I \to \mathbb{R}^3$, $\hat{\sigma}^e(s,t) = A(t) \cdot e(s) + b(t)$.

Parametrization

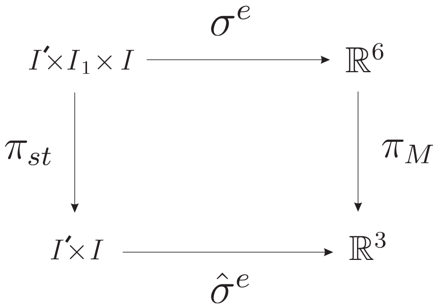


Figure: The above diagram commutes.

Parametrization

- $\pi_{st}(\mathcal{F})$ serves as a parametrization space for contact set C

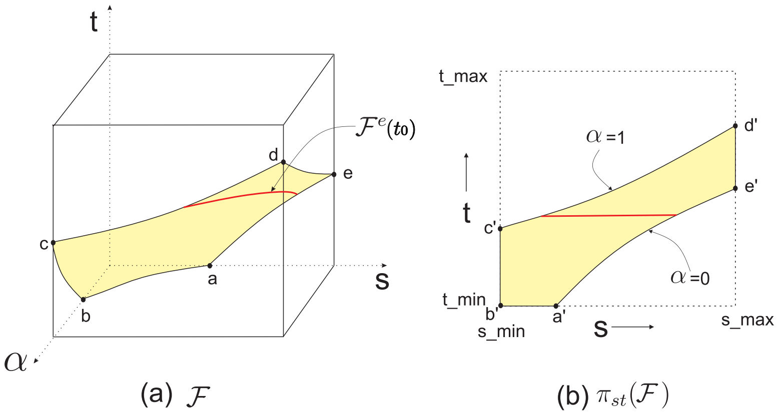


Figure: The funnel $\mathcal{F}$ and $\pi_{st}(\mathcal{F})$.

- $\partial C = \hat{\sigma}^e(\pi_{st}(\mathcal{F} \cap \partial(I' \times I_1 \times I)))$.

Sweeping **sharp** solids

A vertex will trace edges and an edge will trace faces

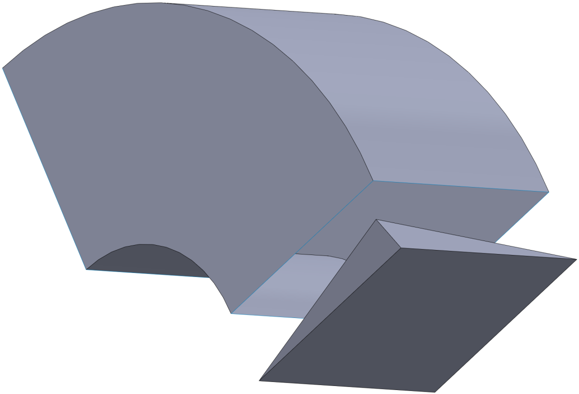


Figure: A pyramid swept along a curvilinear trajectory

Sweeping **sharp** solids

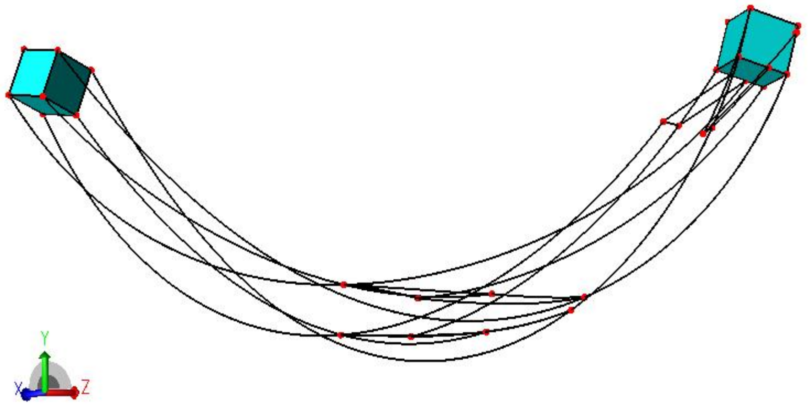


Figure: The 1-cage of the envelope obtained by sweeping a cube.

Thank You

धन्यवाद