Sweep Surfaces for CAGD

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Outline of the work

- When introducing a new surface type in a CAD kernel
  - Parametrization: Local aspects
  - Topology: Global aspects
  - Self-intersection: Global aspects
- Parametrization: Funnel
- Self-intersection: Trim curves and locus of $\theta = 0$
- Topology: Local homeomorphism between solid and envelope.
- Further, sweeping sharp solids.
A coin is translated along a parabolic trajectory in 2-D. At each time instance $t$, there are two points-of-contact.
A non-decomposable 2-D sweep

A coin is translated along a higher-curvature parabolic trajectory. What is the envelope in this case?

Figure: A ‘non-decomposable’ 2-D sweep
A non-decomposable 2-D sweep

A coin is translated along a higher-curvature parabolic trajectory. What is the envelope in this case? The parts connecting the green point to the endpoints of the red-curve also need to be trimmed to construct the correct envelope!
- Brep: A solid $M$ in $\mathbb{R}^3$ represented by its boundary
- A trajectory in the group of rigid motions:
  \[ h : \mathbb{R} \rightarrow (SO(3), \mathbb{R}^3), \ h(t) = (A(t), b(t)) \] where
  \[ A(t) \in SO(3), \ b(t) \in \mathbb{R}^3, \ t \in I \]
- Action of $h$ on $M$ at time $t$:
  \[ M(t) = \{ A(t) \cdot x + b(t) | x \in M \} \]
- Trajectory of a point $x$:
  \[ \gamma_x : I \rightarrow \mathbb{R}^3, \ \gamma_x(t) = A(t) \cdot x + b(t) \]
Envelope Definition

- **Swept volume** $\mathcal{V} := \bigcup_{t \in I} M(t)$.
- **Envelope** $\mathcal{E} := \partial \mathcal{V}$.
- **Correspondence** $R = \{ (y, x, t) \in \mathcal{E} \times M \times I | y = \gamma_x(t) \}$.
- $R \subset \mathcal{E} \times \partial M \times I$.
- $\partial M$ induces the brep structure on $\mathcal{E}$ via $R$. 
Outward normal to $\partial M$ at $x$: $N(x)$.

Velocity of $\gamma_x(t)$: $\gamma_x'(t)$.

Define $g : \partial M \times I \rightarrow \mathbb{R}$ as $g(x, t) = \langle A(t) \cdot N(x), \gamma_x'(t) \rangle$.

For $I = [t_0, t_1]$, $\gamma_x(t) \in \mathcal{E}$ only if:

(i) $g(x, t) = 0$, or
(ii) $t = t_0$ and $g(x, t) \leq 0$, or
(iii) $t = t_1$ and $g(x, t) \geq 0$. 
- **Curve of contact** at $t_0 \in I$:
  \[ C(t_0) = \{ \gamma_x(t_0) | x \in \partial M, g(x, t_0) = 0 \} \].
- **Contact set** $C = \bigcup_{t \in I} C(t)$. 

![Diagram showing curve of contact at t=0 and contact-set](image.png)
Parametrizations: Faces

- Smooth/regular surface $S$ underlying face $F$ of $\partial M$; $u, v$: parameters of $S$.
- Sweep map $\sigma : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^3$
  \[ \sigma(u, v, t) = A(t) \cdot S(u, v) + b(t) \]
- For sweep interval $I = [t_0, t_1]$, we define the following subsets of the parameter space
  \[ \mathcal{L} = \{(u, v, t_0) \in \mathbb{R}^2 \times \{t_0\} \text{ such that } f(u, v, t_0) \leq 0\} \]
  \[ \mathcal{F} = \{(u, v, t) \in \mathbb{R}^2 \times I \text{ such that } f(u, v, t) = 0\} \]
  \[ \mathcal{R} = \{(u, v, t_1) \in \mathbb{R}^2 \times \{t_1\} \text{ such that } f(u, v, t_1) \geq 0\} \]
- $C = \sigma(\mathcal{F})$
Parametrizations
\[ C = \sigma(\mathcal{F}) \]

Figure: The funnel and the contact-set.
For $t_0 \in I$, $R_{t_0} := \{(y, x, t) \in R | t = t_0\}$.

- Projections $\tau : R \rightarrow I$ and $Y : R \rightarrow E$ as $\tau(y, x, t) = t$ and $Y(y, x, t) = y$.

- Sweep $(M, h, I)$ is simple if for all $t \in I^o$, $C(t) = Y(R_t)$.

- No trimming needed: $E = \sigma(\mathcal{L} \cup \mathcal{F} \cup \mathcal{R})$. 
Self-intersections
Trim set: Not all sweeps are simple

- **Trim set** $T := \{x \in C | \exists t \in I, x \in M^o(t)\}$.
- **p-trim set** $pT := \sigma^{-1}(T) \cap \mathcal{F}$.
- Clearly, $T \cap \mathcal{E} = \emptyset$.
- Extend the correspondence $R$ to $C \times M \times I$:
  $\tilde{R} := \{(y, x, t) \in C \times M \times I | y = A(t) \cdot x + b(t)\}$.
- $\tilde{R} \not\subset C \times \partial M \times I$
- **Trim curve** $\partial T$: boundary of $\overline{T}$.
- **p-trim curve**: $\partial pT$: boundary of $\overline{pT}$.
- For $p = (u, v, t) \in \mathcal{F}$, let $\sigma(p) = y$. $L : \mathcal{F} \rightarrow 2^{\mathbb{R}}$, $L(p) := \tau(y\tilde{R})$
- Define $\ell : \mathcal{F} \rightarrow \mathbb{R} \cup \infty$, 
  \[
  \ell(p) = \inf_{t' \in L(p) \setminus \{t\}} \| t - t' \| \quad \text{if } L(p) \neq \{t\} \\
  = \infty \quad \text{if } L(p) = \{t\}
  \]
- Define $t_{-\text{sep}} = \inf_{p \in \mathcal{F}} \ell(p)$.
- **Elementary trim curve**: There exists $\delta > 0$ such that for all $p \in C$, $\ell(p) > \delta$.

- **Singular trim curve**: $\inf_{p \in C} \ell(p) = 0$.
Decomposability

- Given $I$, call a partition $\mathcal{P}$ of $I$ into consecutive intervals $I_1, I_2, \ldots, I_{k_{\mathcal{P}}}$ to be of width $\delta$ if $\max\{\text{length}(I_1), \text{length}(I_2), \ldots, \text{length}(I_{k_{\mathcal{P}}})\} = \delta$.

- $(M, h, I)$ is **decomposable** if there exists $\delta > 0$ such that for all partitions $\mathcal{P}$ of $I$ of width $\delta$, each sweep $(M, h, I_i)$ is simple for $i = 1, \ldots, k_{\mathcal{P}}$.

- The sweep $(M, h, I)$ is decomposable iff $t-\text{sep} > 0$. Further, if $t-\text{sep} > 0$ then all the p-trim curves are elementary.

(a) Decomposable sweep  
(b) Non-decomposable sweep
For $p \in \mathcal{F}$, $\{\sigma_u(p), \sigma_v(p), \sigma_t(p)\}$ are l.d.

Let $\sigma_t(p) = n(p).\sigma_u(p) + m(p).\sigma_v(p)$, $n$ and $m$ continuous on $\mathcal{F}$.

Define $\theta : \mathcal{F} \to \mathbb{R}$,

$$\theta(p) = n(p) \cdot f_u(p) + m(p) \cdot f_v(p) - f_t(p)$$

If for all $p \in \mathcal{F}$, $\theta(p) > 0$, then the sweep is decomposable. If there exists $p \in \mathcal{F}$ such that $\theta(p) < 0$, then the sweep is non-decomposable.

$\theta$ invariant of the parametrization of $\partial M$.

Arises out of relation between two 2-frames on $\mathcal{T}_C$.

Is a non-singular function.
A geometric invariant on $\mathcal{F}$

- $\theta$ partitions the $\mathcal{F}$ into (i) $\mathcal{F}^+ := \{ p \in \mathcal{F} | \theta(p) > 0 \}$, (ii) $\mathcal{F}^- := \{ p \in \mathcal{F} | \theta(p) < 0 \}$ and (iii) $\mathcal{F}^0 := \{ p \in \mathcal{F} | \theta(p) = 0 \}$.
- Define $C^+ := \sigma(\mathcal{F}^+)$, $C^- := \sigma(\mathcal{F}^-)$ and $C^0 := \sigma(\mathcal{F}^0)$.

- $C^- \subset T$.
- $C^0$: The set of points where $\text{dim}(T_C) < 2$. 
Trimming non-decomposable sweeps

Figure: Example of a non-decomposable sweep: an elliptical cylinder being swept along $y$-axis while undergoing rotation about $z$-axis. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where $\theta$ is negative is shown in green.
• If \( c \) is a singular p-trim curve and \( p_0 \in c \) is a limit-point of \( (p_n) \subset c \) such that \( \lim_{n \to \infty} \ell(p_n) = 0 \), then \( \theta(p_0) = 0 \).

• **singular trim point**: A limit point \( p \) of a singular p-trim curve \( c \) such that \( \theta(p) = 0 \).

• Every curve \( c \) of \( \partial pT \) has a curve \( F^0_c \) of \( F^0 \) which makes contact with it.

• \( F^0 \) is easy to compute since \( \nabla \theta \) is non-zero.
Let $\Omega$ be a parametrization of a curve $F^0_i$ of $F^0$. Let $\Omega(s_0) = p_0 \in F^0_i$ and $\bar{z} := (n, m, -1) \in \text{null}(J_\sigma)$ at $p_0$, i.e., $n\sigma_u + m\sigma_v = \sigma_t$. Define the function $\varrho : F^0 \to \mathbb{R}$ as follows.

$$\varrho(s_0) = \left\langle \bar{z} \times \frac{d\Omega}{ds}|_{s_0}, \nabla f|_{p_0} \right\rangle$$

$\varrho$ is a measure of the oriented angle between the tangent at $p_0$ to $F^0_i$ and the kernel (line) of the Jacobian $J_\sigma$ restricted to the tangent space $T_{\bar{F}}(p_0)$.

If $p_0$ is a singular trim point, then $\varrho(p_0) = 0$. 
Examples of non-decomposable sweeps

Figure: Example of a non-decomposable sweep: a cone being swept along a parabola. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where $\theta$ is negative is shown in green.
Examples of non-decomposable sweeps

Figure: Example of a non-decomposable sweep: a cylinder being swept along a cosine curve in $xy$-plane while undergoing rotation about $x$-axis. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where $\theta$ is negative is shown in green.
Examples of non-decomposable sweeps

Figure: Example of a non-decomposable sweep: a blended intersection of a sphere and an ellipsoid being swept along a circular arc in $xy$-plane while undergoing rotation about $z$-axis. The curve $\theta = 0$ is shown in red and trim curve is shown in blue. The portion of the swept edges where $\theta$ is negative is shown in green.
Nested trim curves

Figure: A singular p-trim curve nested inside an elementary p-trim curve
Topology
Assume w.l.o.g. \((M, h, I)\) is simple.

Let \(F\) be a face of \(\partial M\) and \(C^F\) be its contact set.

The correspondence \(R\) induces the natural map \(\pi : C^F \rightarrow F\)

\[ \pi(y) = x \text{ such that } (y, x, t) \in R. \]

\(\pi\) is a well defined map.

For \(p \in F^F\), let \(\sigma(p) = y. \) \(\pi\) is a local homeomorphism at \(y\) if \(f_t(p) \neq 0.\)

Proof. \(\pi'\) is a local homeomorphism.

![Diagram](image)

Figure: The above diagram commutes.
Orientability of the envelope

(a) $\partial M$

(b) Contact set
When is $\pi$ orientation preserving/reversing?

- For $p \in \mathcal{F}$ let $\sigma(p) = y$ and suppose $f_t(p) \neq 0$.
- $\pi$ is orientation preserving/reversing at $y$ if $-\frac{\theta(p)}{f_t(p)}$ is positive/negative respectively.
- $-\frac{\theta}{f_t}$ is a geometric invariant.

Figure: In the above example, $\pi(y) = x$. The map $\pi$ is orientation preserving at $y$. 
When is $\pi$ orientation preserving/reversing?

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Define the following subsets of a nbhd. \( M \subset F(t_0) \) of a point \( y \in C(t_0) \)

\[
\begin{align*}
    f^+ &= \{ q \in M | f(\sigma^{-1}(q)) > 0 \} \\
    f^0 &= \{ q \in M | f(\sigma^{-1}(q)) = 0 \} = C(t_0) \cap M \\
    f^- &= \{ q \in M | f(\sigma^{-1}(q)) < 0 \}
\end{align*}
\]

Figure: Positive and negative hemispheres at a point \( y \in \partial M(t_0) \).
Geometric meaning of $-\frac{\theta}{f}$.

- **Contributing curve** at $t_0$ for $t$ is defined as the set $\{\gamma_x(t_0)|x \in \partial M, g(x, t) = 0\}$ and denoted by $t_0C(t)$.
- $t_0C(t_0) = C(t_0)$
Geometric meaning of $-\frac{\theta}{f_t}$

Figure: The map $\pi$ is orientation preserving (a) The curves $t_0C(t)$ are plotted on $\partial M(t_0)$ at time instances $t_1 < t_2 < t_3$. The vector $J_\sigma \cdot \alpha$ is plotted at few points. (b) The curves $C(t)$ are plotted on $C$ at time instances $t_1 < t_2 < t_3$. 
Geometric meaning of $-\frac{\theta}{f_t}$

Figure: The map $\pi$ is orientation reversing (a) The curves $t_0C(t)$ are plotted on $\partial M(t_0)$ at time instances $t_1 < t_2 < t_3$. The vector $J_\sigma \cdot \alpha$ is plotted at few points. (b) The curves $C(t)$ are plotted on $C$ at time instances $t_1 < t_2 < t_3$. 
Geometric meaning of $-\theta/f_t$

Figure: The map $\pi$ is orientation preserving in a neighborhood of the point $C^{v_1}$ and reversing in a neighborhood of the point $C^{v_2}$. (a) The curves $t_0C(t)$ are plotted on $\partial M(t_0)$ at time instances $t_1 < t_2 < t_3 < t_4 < t_5$. The vector $J_\sigma \cdot \alpha$ is plotted at few points. (b) The curves $C(t)$ are plotted on $C$ at time instances $t_1 < t_2 < t_3 < t_4 < t_5$. 
Orienting edges of $\mathcal{E}$

Figure: Orienting $C^e$. In this case $-\frac{\theta^F}{f_t^F}$ is negative at the point $y$. 
Orienting edges of $\mathcal{E}$

Figure: Edges in parameter space $(s, t)$, generated by an edge $e \in \partial M$. 
Computing adjacencies

- If faces $C^F$ and $C^{F'}$ are adjacent in $C$ then the faces $F$ and $F'$ are adjacent in $\partial M$.
- If edges $C^e$ and $C^{e'}$ are adjacent in $C$ then $e$ and $e'$ are adjacent in $\partial M$.
- If an edge $C^e$ bounds a face $C^F$ in $C$ then the edge $e$ bounds the face $F$ in $\partial M$.
- If a vertex $C^z$ bounds an edge $C^e$ in $C$ then the vertex $z$ bounds the edge $e$ in $\partial M$.
- The unit outward normal varies continuously across adjacent geometric entities in $C$. 
Figure: A simple bottle being swept along a screw motion with compounded rotation. Correspondence between faces of $\partial M$ and those of the envelope is shown by color coding.
Simple sweep examples
Simple sweep examples
Overall computational framework

**Algorithm 1 Solid sweep**

```plaintext
for all $F$ in $\partial M$ do
    for all $e$ in $\partial F$ do
        for all $z$ in $\partial e$ do
            Compute vertices $C^z$ generated by $z$
        end for
        Compute edges $C^e$ generated by $e$
        Orient edges $C^e$
    end for
    Compute $C^F(t_0)$ and $C^F(t_1)$
    Compute loops bounding faces $C^F$ generated by $F$
    Compute faces $C^F$ generated by $F$
    Orient faces $C^F$
end for
for all $F_i, F_j$ adjacent in $\partial M$ do
    Compute adjacencies between faces in $C^{F_i}$ and $C^{F_j}$
end for
```
How topology of $C(t)$ varies

- $t : \mathcal{F} \rightarrow \mathbb{R}, (u, v, t) \mapsto t$ is a Morse function.
- Critical points of this function.

Figure: Number of connected components of $C(t)$ is 1, 2 and 1 for $t \in (0, t_1), (t_1, t_2)$ and $(t_2, 1)$ respectively.
How topology of $C(t)$ varies

Figure: Number of connected components of $C(t)$ varies from 1 to 2 to 1 with time.
Sweeping \textit{sharp} solids
Sweeping \textbf{sharp} solids

Figure: A G1-discontinuous solid.
Cone of normals and Cone bundle

For a point \( x \in \bigcap_{i=1}^{n} F_i \), define the **cone of normals** at \( x \) as

\[
\mathcal{N}_x = \left\{ \sum_{i=1}^{n} \alpha_i \cdot N_i(x) \right\},
\]

where, \( N_i(x) \) is the unit outward normal to face \( F_i \) at point \( x \) and \( \alpha_i \in \mathbb{R}, \alpha_i \geq 0 \) for \( i = 1, \ldots, n \) and \( \sum_{i=1}^{n} \alpha_i = 1 \).

For a subset \( X \) of \( \partial M \), the **cone bundle** is defined as the disjoint union of the cones of normals at each point in \( X \) and denoted by \( \mathcal{N}_X \), i.e.,

\[
\mathcal{N}_X = \bigsqcup_{x \in X} \mathcal{N}_x = \bigcup_{x \in X} \{(x, N(x)) | N(x) \in \mathcal{N}_x\}.
\]
Figure: A solid and its cone bundle.
For \((x, N(x)) \in \mathbf{N}_{\partial M}\) and \(t \in I\), define the function \(g : \mathbf{N}_{\partial M} \times I \rightarrow \mathbb{R}\) as
\[
g(x, N(x), t) = \langle A(t) \cdot N(x), v_x(t) \rangle
\]
For \((y, x, t) \in R\) and \(I = [t_0, t_1]\), either
(i) \(t = t_0\) and there exists \(N(x) \in \mathcal{N}_x\) such that \(g(x, N(x), t) \leq 0\) or
(ii) \(t = t_1\) and there exists \(N(x) \in \mathcal{N}_x\) such that \(g(x, N(x), t) \geq 0\) or
(iii) There exists \(N(x) \in \mathcal{N}_x\) such that \(g(x, N(x), t) = 0\).

Projection \(\pi_M : \mathbf{N}_{\partial M} \rightarrow \partial M\) as \(\pi_M(x, N(x)) = x\).
Necessary condition

- **Normals of contact at** \( t_0 \)
  \[ C(t_0) := \{ (\gamma_x(t_0), A(t_0) \cdot N(x)) \in N_{\partial M}(t_0) | g(x, N(x), t_0) = 0 \}. \]

- **Curve of contact at** \( t_0 \)
  \[ C(t_0) := \pi_M(C(t_0)). \]

![Diagram](image.png)
For $x$ in edge $E = F_1 \cap F_2$, parametrize $N_x$ with $\alpha \in [0, 1]$ as
\[ N_x(\alpha) = \alpha \cdot N_1(x) + (1 - \alpha) \cdot N_2(x) \]

Let $I'$ be the domain of curve $e$ underlying edge $E$.

Define function $f$ on the parameter space $I' \times I_1 \times I$ to $\mathbb{R}$ as
\[ f(s, \alpha, t) = g(e(s), N_e(s)(\alpha), t). \]

Funnel $\mathcal{F} = \{(s, \alpha, t) \in I' \times I_1 \times I \text{ such that } f(s, \alpha, t) = 0\}$

Sweep map $\sigma^e : I' \times I_1 \times I \rightarrow \mathbb{R}^6$ is defined as
\[ \sigma^e(s, \alpha, t) = (\gamma_{e(s)}(t), A(t) \cdot N_{e(s)}(\alpha)) \]

Projection $\pi_{st} : I' \times I_1 \times I \rightarrow I' \times I$, $\pi_{st}(s, \alpha, t) = (s, t)$.

Projected sweep map $\hat{\sigma}^e : I' \times I \rightarrow \mathbb{R}^3$, $\hat{\sigma}^e(s, t) = A(t) \cdot e(s) + b(t)$.
Figure: The above diagram commutes.
- $\pi_{st}(\mathcal{F})$ serves as a parametrization space for contact set $C$

![Diagram](image)

(a) $\mathcal{F}$

(b) $\pi_{st}(\mathcal{F})$

Figure: The funnel $\mathcal{F}$ and $\pi_{st}(\mathcal{F})$.

- $\partial C = \hat{\sigma}^e(\pi_{st}(\mathcal{F} \cap \partial(l' \times l_1 \times l)))$. 
A vertex will trace edges and an edge will trace faces.

Figure: A pyramid swept along a curvilinear trajectory.
Figure: The 1-cage of the envelope obtained by sweeping a cube.
Thank You