# CS618: Program Analysis
# Assignment 3 (Solutions)

**Q1.** Let $(S, \wedge)$ be a semilattice. Let $f : S \to S$ be a function. Prove that the following two definitions for *monotonicity* of $f$ are equivalent:

$$\forall x, y \in S : f(x \wedge y) \leq f(x) \wedge f(y)$$

and

$$\forall x, y \in S : x \leq y \Rightarrow f(x) \leq f(y)$$

- $\Rightarrow$: Let $x \leq y$. Then:
  $x \wedge y = x \;\Rightarrow\; f(x \wedge y) = f(x)$
  $\Rightarrow\; f(x) \leq f(x) \wedge f(y)$
  $\Rightarrow\; f(x) \leq f(y)$ // Recall properties of $\wedge$
- $\Leftarrow$:
  $x \wedge y \leq x \;\Rightarrow\; f(x \wedge y) \leq f(x)$
  $x \wedge y \leq y \;\Rightarrow\; f(x \wedge y) \leq f(y)$
  $\Rightarrow\; f(x \wedge y)$ is a lower bound on $f(x)$ and $f(y)$
  $\Rightarrow\; f(x \wedge y) \leq f(x) \wedge f(y)$

**Q2. Knaster-Tarski Fixed Point Theorem**: Let $f : S \to S$ be a monotonic function on a complete lattice $(S, \vee, \wedge)$. Using the concepts covered in class, prove that fix-points of $f$ (i.e. members of fix$(f)$) form a complete lattice.

Let $Y$ be an arbitrary subset of fix$(f)$. We need to show that lub$(Y)$ and glb $(Y)$ are in fix$(f)$. We show lub$(Y) \in$ fix$(f)$. Ther other part is similar.

Consider $y = \wedge Y$, and the set $X = \{x \mid x \in S, x \leq y\}$.

$(S, \vee, \wedge)$ is a complete lattice $\Rightarrow$ $y \in S$ ........ (1)
$y \leq y \Rightarrow y \in X$ ........ (2)
(1) and (2) $\Rightarrow$ $y$ is lub$(X, \vee, \wedge)$ ........ (3)
glb$(S, \vee, \wedge))$ is also glb$(X, \vee, \wedge)$ ........ (4)
$X \subseteq S$ ........ (5)
(3), (4) and (5) $\Rightarrow$ $(X, \vee, \wedge)$ is a complete lattice ........ (6)

Consider a restriction of $f$ over $X$. We show that $f : X \to X$. Consider $x \in X$ and $z \in Y$.

$y = \wedge Y \Rightarrow x \leq y \leq z$ ........ (7)
(7) $\Rightarrow f(x) \leq f(z)$ ........ (8)
$z$ is a fixed-pt of $f$ $\Rightarrow f(z) = z$ ........ (9)
(8) and (9) $\Rightarrow f(x) \leq z$ ........ (10)
Since $z$ is an arbitrary elememt in $Y$, and $y = \wedge Y$
$\Rightarrow f(x) \leq y$ ........ (11)
$f(x) \in S$ and (11) $\Rightarrow f(x) \in X$ ........ (12)
Thus, $x \in X \Rightarrow f(x) \in X$. In other words, $f : X \to X$.

$(X, \vee, \wedge)$ is a complete lattice (6), $f : X \to X$ is a monotonic function over $X$. Therefore, by other statements of Knaster-Tarski theorem, $f$ has a *greatest fixed-point* $z$ in $X$. But then, $z \in \text{fix}(f), z \leq y = \wedge Y$ implying $\text{lub}(Y) = z \in \text{fix}(f)$

**Q3.** An edge in a flow graph is a back edge if its head dominates its tail. Prove that every back edge is a retreating edge in every DFST of every flow graph.

Let $m \to n$ be a back edge. Then, $n$ dominates $m$. Any path from ENTRY to $m$ must go through $n$. No matter how we construct DFST, $n$ will be touched before $m$. Thus, DFS number of $n$ will be less than DFS number of $m$ in any DFST $\Rightarrow m \to n$ will be a retreating edge in any DFST.

**Q4.** In a flow graph, the natural loop of a back edge $a \to b$ is $\{b\}$ plus the set of nodes that can reach $a$ without going through $b$. Prove that two natural loops are either disjoint, identical, or nested.

**Notation:** $n_1 \overset{\overline{n_2}}{\rightsquigarrow} n_3$ denotes a path from $n_1$ to $n_3$ that does not go through $n_2$. $n_1 \overset{n_2}{\rightsquigarrow} n_3$ denotes a path from $n_1$ to $n_3$ that goes through $n_2$

**Lemma A:** If node $x$ is in the natural loop of a back edge $a \to b$, then $b$ dominates $x$.

**Proof Hint:** Consider the path ENTRY $\rightsquigarrow x \overset{\overline{b}}{\rightsquigarrow} a$, and note that $b$ dominates $a$.

**Proof of main statement**: Assume to the contrary that natural loops for back edges $a_1 \to b_1$ (say, $L_1$) and $a_2 \to b_2$ (say, $L_2$) intersect, but are niether identical, nor nested. Thus, there is a node $x$ that belongs to both $L_1$ and $L_2$, a node $y_1$ that belongs to $L_1$ but not $L_2$, and a node $y_2$ that belongs to $L_2$ but not $L_1$.

Clearly both $b_1$ and $b_2$ dominate $x$ (From Lemma above). Thus, either $b_1$ dominates $b_2$ or $b_2$ dominates $b_1$ (Proof?). Without loss of generality, assume $b_1$ dominates $b_2$. Clearly, $b_2$ can not dominate $b_1$ so $b_1$ does not belong to $L_2$ implying that paths from $b_1$ to $a_2$, if any, must go through $b_2$.

Now consider the path $y_2 \overset{\overline{b_1}}{\rightsquigarrow} a_2 \to b_2 \overset{\overline{b_1}}{\rightsquigarrow} x \overset{\overline{b_1}}{\rightsquigarrow} a_1$: such a path must exist (why?), and does not involve $b_1$. But then, $y_2$ belongs to $L_1$ - a contradiction.