CS618: Program Analysis
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Simply Typed Lambda Calculus

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Reference Book

Types and Programming Languages by Benjamin C. Pierce

Simple Types over Bool

T :=		– Types
	Bool	– Boolean Type
	$T \rightarrow T$	– Function Type

type constructor $\rightarrow$ is right-associative, i.e., $T_1 \rightarrow T_2 \rightarrow T_3$ stands for $T_1 \rightarrow (T_2 \rightarrow T_3)$

Examples

For each of the type below, write a function (in your favourite programming language) that has the required type:

- Bool $\rightarrow$ Bool
- Bool $\rightarrow$ Bool $\rightarrow$ Bool
- (Bool $\rightarrow$ Bool) $\rightarrow$ Bool
- (Bool $\rightarrow$ Bool) $\rightarrow$ Bool $\rightarrow$ Bool
- (Bool $\rightarrow$ Bool $\rightarrow$ Bool) $\rightarrow$ Bool
- (Bool $\rightarrow$ Bool $\rightarrow$ Bool) $\rightarrow$ Bool
- ((Bool $\rightarrow$ Bool) $\rightarrow$ Bool) $\rightarrow$ Bool

The Abstract Syntax

Simply Typed λ-terms with conditions and Booleans

t	:=	x	*– Variable*
	|	$\lambda x : T.\, \mathrm{t}$	*– Abstraction*
	|	t t	*– Application*
	|	`true`	*– constant true*
	|	`false`	*– constant false*
	|	`if` t `then` t `else` t	*– conditional*

Recap: The Set of Values

v	:=		*– values*
		$\lambda x : T.\, \mathrm{t}$	*– Abstraction Value*
	|	`true`	*– value true*
	|	`false`	*– value false*

Evaluation

$$\frac{\mathrm{t}_1 \to \mathrm{t}_1'}{\mathrm{t}_1\ \mathrm{t}_2 \to \mathrm{t}_1'\ \mathrm{t}_2} \qquad \text{(E-APP1)}$$

$$\frac{\mathrm{t}_2 \to \mathrm{t}_2'}{v\ \mathrm{t}_2 \to v\ \mathrm{t}_2'} \qquad \text{(E-APP2)}$$

$$(\lambda x : T_1.\, \mathrm{t}_1) v_2 \to [x \mapsto v_2]\mathrm{t}_1 \qquad \text{(E-APPABS)}$$

The Typing Relation

- A *Typing Context* or *Type Environment*, Γ, is a sequence of variables with their types
- $\Gamma, x : T$ denotes extending Γ with a new variable x having type T
  - The name x is assumed to be distinct from any existing names in Γ

The Typing Relation

$$\frac{\Gamma, x : T_1 \vdash \mathrm{t}_2 : T_2}{\Gamma \vdash \lambda x : T_1.\, \mathrm{t}_2 : T_1 \to T_2} \quad \text{(T-ABS)}$$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \text{(T-VAR)}$$

$$\frac{\Gamma \vdash \mathrm{t}_1 : T_1 \to T_2 \qquad \Gamma \vdash \mathrm{t}_2 : T_1}{\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : T_2} \quad \text{(T-APP)}$$

Inversion of the Typing Relation

- If $\Gamma \vdash x : R$, then $x : R \in \Gamma$.
- If $\Gamma \vdash \lambda x : T_1.\, \mathrm{t}_2 : R$, then $R = T_1 \to R_2$ for some R_2 with $\Gamma, x : T_1 \vdash \mathrm{t}_2 : R_2$.
- If $\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : R$, then $\exists T_1\ s.t.\ \Gamma \vdash \mathrm{t}_1 : T_1 \to R$ and $\Gamma \vdash \mathrm{t}_2 : T_1$.
- If $\Gamma \vdash \mathtt{true} : R$, then $R =$ Bool.
- If $\Gamma \vdash \mathtt{false} : R$, then $R =$ Bool.
- If $\Gamma \vdash \mathtt{if}\ \mathrm{t}_1\ \mathtt{then}\ \mathrm{t}_2\ \mathtt{else}\ \mathrm{t}_3 : R$, then
  - $\Gamma \vdash \mathrm{t}_1 :$ Bool
  - $\Gamma \vdash \mathrm{t}_2 : R$
  - $\Gamma \vdash \mathrm{t}_3 : R$

Exercises

- For each of the term t below, find context Γ and type T such that

$$\Gamma \vdash \mathrm{t} : T$$

  - t is $\lambda x.\, x$
  - t is ($(x\ z)$ $(y\ z)$)
  - t is $\lambda y.\, x$
  - t is $x\ x$

Uniqueness of Types

- In a given type context Γ, A term t, such that the free variables of t are in Γ, has at most one type.
- If t is typeable, then its type is unique.
- Moreover, there is just one derivation of this typing built from the inference rules.

Some Properties

- **Permutation:** If $\Gamma \vdash \mathsf{t} : T$ and Δ is a permutation of Γ, then $\Delta \vdash \mathsf{t} : T$.
  - The derivation with Δ has the same depth as the derivation with Γ.
- **Weakening:** If $\Gamma \vdash \mathsf{t} : T$ and $x \notin \text{domain}(\Gamma)$, then $\Gamma, x : S \vdash \mathsf{t} : T$.
  - The derivation with $\Gamma, x : S$ has the same depth as the derivation with Γ.

Progress

- **Progress:** A well-typed term is not stuck.
  - If $\vdash \mathsf{t} : T$, then t is either a value or there exists some t' such that $\mathsf{t} \rightarrow \mathsf{t}'$.

Preservation

- **Preservation of Types under Substitution:** If $\Gamma, x : S \vdash \mathsf{t} : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]\mathsf{t} : T$.
- **Preservation:** If a well-typed term takes a step of evaluation, then the resulting term is also well-typed.
  - If $\Gamma \vdash \mathsf{t} : T$ and $\mathsf{t} \rightarrow \mathsf{t}'$, then $\Gamma \vdash \mathsf{t}' : T$.