Expressiveness for Real-Time Logics

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AVERTS
December 2015

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Metric Temporal Logic [Koymans; de Roever; Pnueli $\sim$ 1990]

MTL = LTL + Timing Constraints

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We consider MTL with integer and rational constants respectively.

Predicate Logic as a Yardstick

Work on real time metric formalisms still does not converge toward a main formalism [. . .] The most natural language to discuss systems that evolve in time is classical predicate logic [. . .] How are we to choose the correct logic without the yardstick of the standard predicate logic?

Hirshfeld and Rabinovich 2004.

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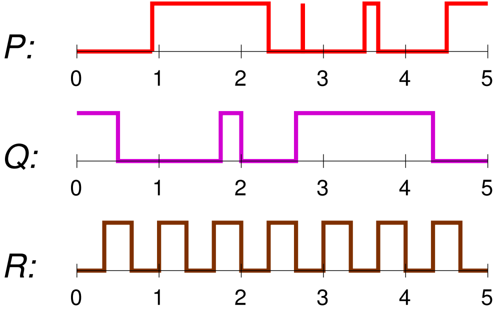

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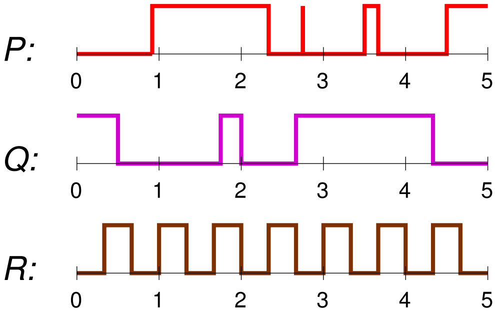

- View signals as expansions of $(\mathbb{R}, <, +1)$.

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Counting modality $C_2(\varphi)$

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Question

Is there an analogue of Kamp's Theorem in the metric setting?

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MTL with integer constants is strictly less expressive than FO$(<, +1)$ *over* $\mathbb{R}$*. Moreover no temporal logic with modalities defined by FO*$(<, +1)$*-formulas of bounded quantifier depth is expressively complete.*

Metric Version of Kamp's Theorem

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Neither theorem contradicts the inexpressiveness result of Hirshfeld and Rabinovich!

Bounded formulas

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Proof.

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Proof.

- Suffices to express all N-bounded $\text{FO}(<, +1)$-formulas.
- Suffices in turn to express all unit formulas of $\text{FO}(<, +1)$.

□

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In MTL with rational constants:

$$\begin{aligned}\varphi \quad = \quad & \mathbf{F}_{(0,\frac{1}{2})} \left(\mathrm{P} \wedge \mathbf{F}_{(0,\frac{1}{2})} \; \mathrm{P}\right) \quad \vee \\ & \left(\mathbf{F}_{(0,\frac{1}{2})} \; \mathrm{P} \wedge \mathbf{F}_{(\frac{1}{2},1)} \; \mathrm{P}\right) \quad \vee \\ & \mathbf{F}_{=1}\left(\mathbf{P}_{(0,\frac{1}{2})} \left(\mathrm{P} \wedge \mathbf{P}_{(0,\frac{1}{2})} \; \mathrm{P}\right)\right)\end{aligned}$$

Decomposition Formulas

A *decomposition formula* $\delta(x, y)$ is any formula of the form

$$\begin{aligned} x < y \wedge \exists z_0 \ldots \exists z_n \, (x = z_0 < \cdots < z_n = y) \\ \wedge \bigwedge \{\varphi_i(z_i) : 0 \leq i < n\} \\ \wedge \bigwedge \{\forall u \, ((z_{i-1} < u < z_i) \to \psi_i(u)) : 0 < i \leq n\} \end{aligned}$$

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Translation of $\delta(x, x+1)$ to MTL is in similar spirit to translation of counting modalities.

Separation of temporal logics

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$$= \mathbf{P}\,\texttt{PEDAL} \;\vee\; (\neg\texttt{BRAKE}\,\mathbf{U}\,\texttt{PEDAL}) \;\vee\; \mathbf{G}(\neg\texttt{BRAKE})$$

Separation of temporal logics

Lemma

LTL is separable.

Theorem (Gabbay 1981)

A temporal logic is expressively complete if and only if it is separable.

Corollary (Kamp's theorem)

LTL is expressively complete.

Quantitative separation

Separation does not hold in the quantitative setting.

For example,

$$\mathbf{G}(\texttt{BRAKE} \rightarrow \mathbf{P}_{(5,10)}\texttt{PEDAL})$$

Quantitative separation

A metric temporal formula is:

- pure distant past if it is invariant on signals that agree on $(-\infty, -1)$
- pure distant future if it is invariant on signals that agree on $(1, \infty)$
- bounded if there is an N such that it is invariant on all signals that agree on $(-N, N)$

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Lemma

MTL is metrically separable.

$$\mathbf{G}(\mathtt{BRAKE} \to \mathbf{P}_{(5,10)}\mathtt{PEDAL}) \quad = \quad \mathbf{G}_{(0,11]}(\mathtt{BRAKE} \to \mathbf{P}_{(5,10)}\mathtt{PEDAL}) \wedge \mathbf{G}_{(11,\infty)}(\mathtt{BRAKE} \to \mathbf{P}_{(5,10)}\mathtt{PEDAL})$$

First Main Result

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Proof.

- Show that MTL can express all bounded formulas.
- For general formulas, give an inductive transformation from FO formulas with one free variable to MTL. For the case $\phi(x) = \exists y\, \psi(x, y)$ use separation and expressive completeness of bounded formulas.

□

Part II

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“Three variables suffice for real-time logic”

Antonopoulos, Hunter, Raza, Worrell (2015)

k-Variable Property

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A class of structures $\mathcal{C}$ has the k-variable property, if every monadic first-order formula with at *most k free variables* is equivalent over $\mathcal{C}$ to a formula with at *most k variables in total*.

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Theorem (Poizat 82; Immerman and Kozen 87)

The class of linear orders has the 3-variable property.

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Temporal logic:

Theorem (Gabbay 81)

A class of posets has finite H-dimension if and only there is a finite expressively complete set of first-order definable temporal modalities.

A Caveat

Theorem (Hodkinson and Simon 97)

For every k there are linear orders with H-dimension k that do not have the ℓ-variable property for any finite ℓ.

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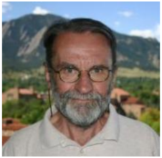
2. Extend the compositional method to the metric setting.

Ehrenfeucht-Fraïssé Games



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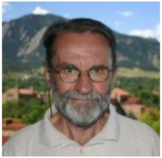 

n-round k-pebble EF game on two signals $\mathcal{A}$ and $\mathcal{B}$:

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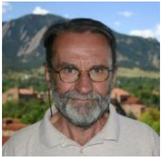 

n-round *k*-pebble EF game on two signals $\mathcal{A}$ and $\mathcal{B}$:

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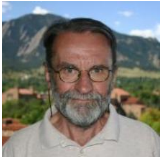

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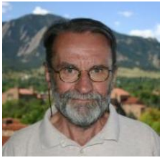 

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- Configuration is a pair of k-tuples $(\overline{a}, \overline{b})$.
- Duplicator wins a play if in each round:
  - $P(a_i)$ iff $P(b_i)$
  - $a_i < a_j$ iff $b_i < b_j$
  - $a_i = a_j + 1$ iff $b_i = b_j + 1$

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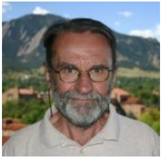

Goal. Show that for all n and k there exists m such that if Duplicator wins the m-round 3-pebble game on any pair of signals $\mathcal{A}$ and $\mathcal{B}$ then she wins the n-round k-pebble game.

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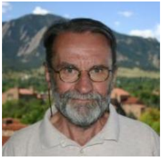 

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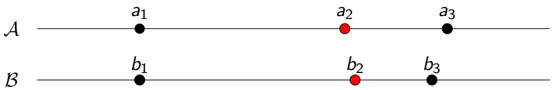
Method. Use composition.

Composition for Linear Orders



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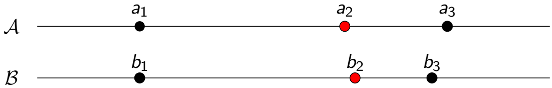

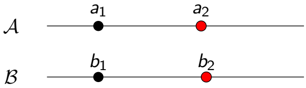

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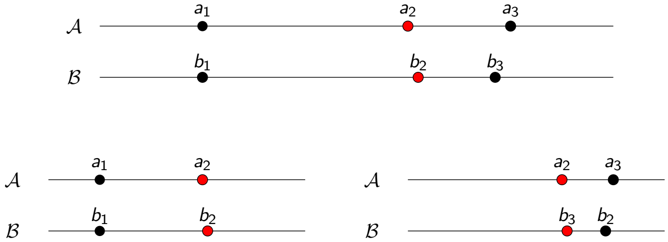

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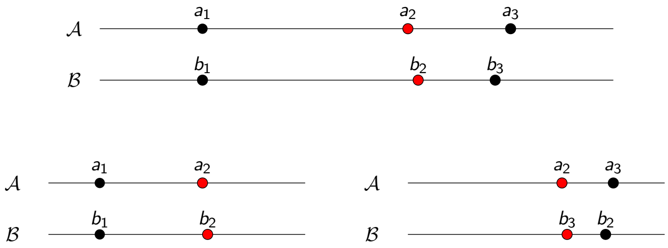

Theorem (Composition Theorem)

If Duplicator wins the n-round games from $(a_1 a_2, b_1 b_2)$ *and* $(a_2 a_3, b_2 b_3)$ *then she wins the n-round game from* $(a_1 a_2 a_3, b_1 b_2 b_3)$.

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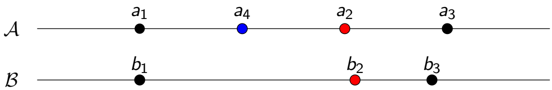

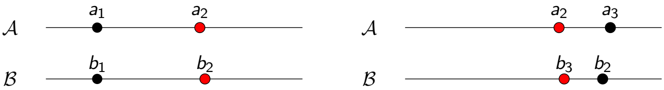

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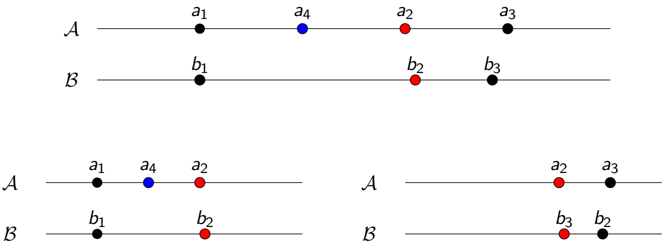

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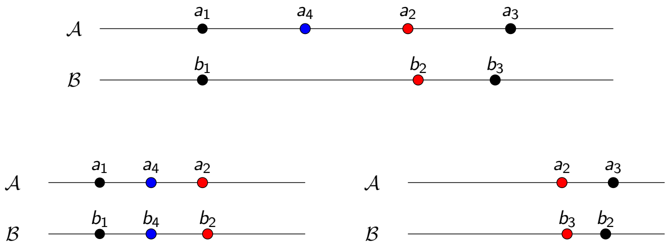

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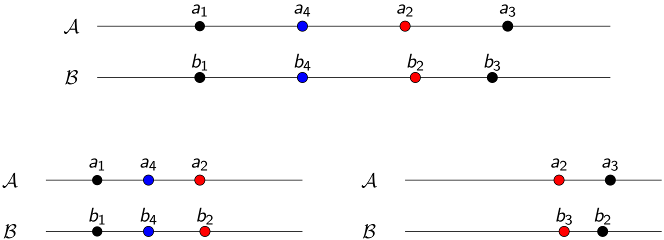

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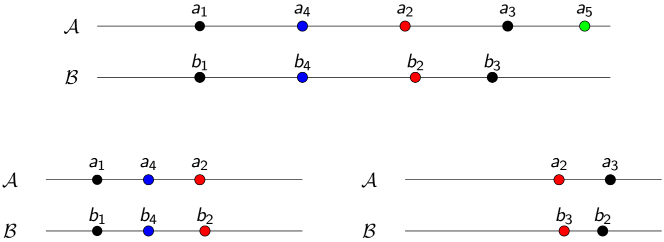

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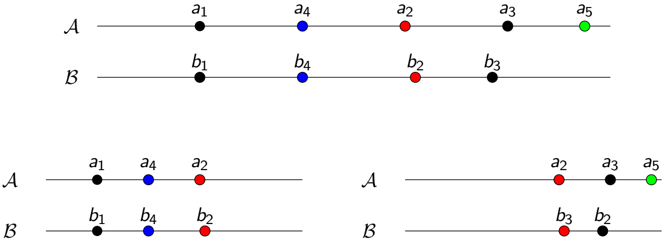
If Duplicator wins the n-round games from $(a_1 a_2, b_1 b_2)$ and $(a_2 a_3, b_2 b_3)$ then she wins the n-round game from $(a_1 a_2 a_3, b_1 b_2 b_3)$.

Composition for Linear Orders



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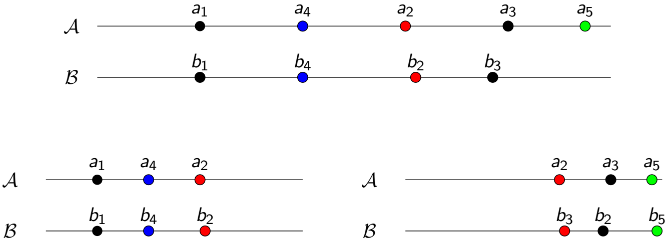

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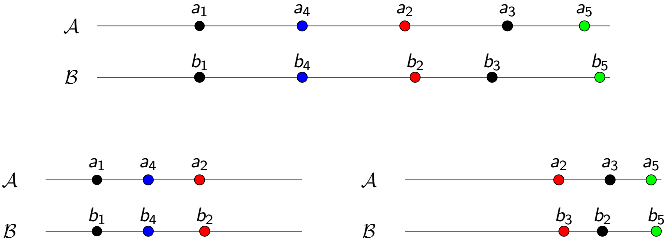

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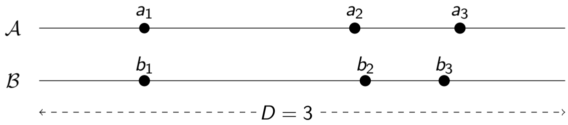

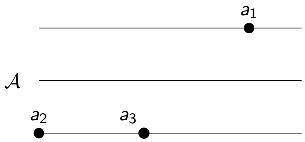

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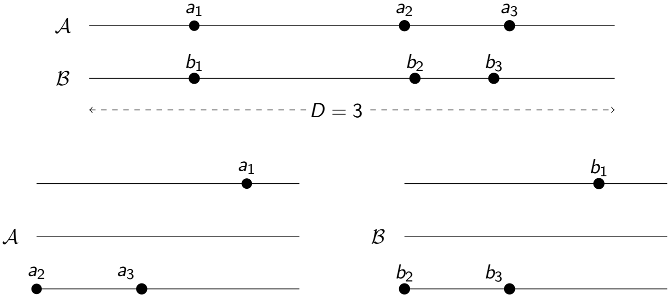

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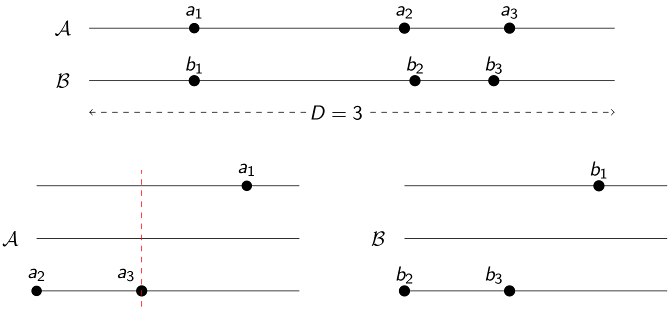

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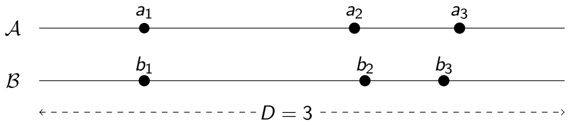

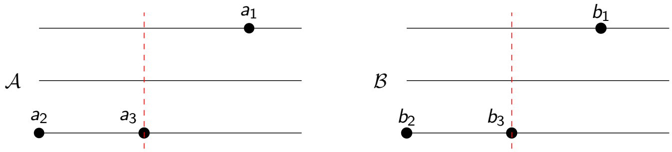

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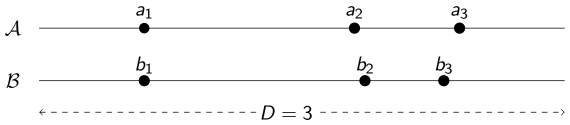
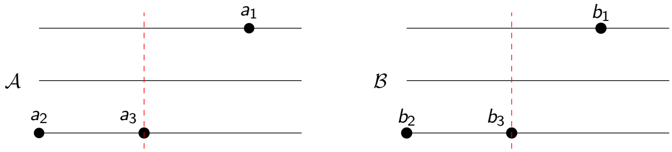




Theorem (D-Local Composition Theorem)

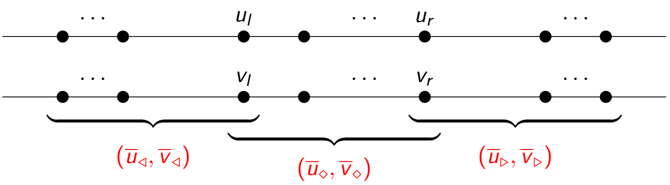
For all n there exists m such that if Duplicator wins the m-round D-local games from $(a_2 a_3, b_2 b_3)$ and $(a_1 a_3, b_1 b_3)$ then she wins the n-round D-local game from $(a_1 a_2 a_3, b_1 b_2 b_3)$.

Local Formulas

Lemma

Let $(\overline{u}, \overline{v})$ *be a 3-configuration of diameter at most* 2^m*. If Spoiler wins the n-round* 2^m*-local game from* $(\overline{u}, \overline{v})$*, then he wins the* $(m+n)$*-round 3-pebble game from* $(\overline{u}, \overline{v})$*.*

Global Composition Theorem



Lemma

Let $D \geq 2^{n+2}$. Suppose that Duplicator wins the $2n$-round D-local game from configuration $(\overline{u}_\diamond, \overline{v}_\diamond)$ and the n-round games from configurations $(\overline{u}_\triangleleft, \overline{v}_\triangleleft)$ and $(\overline{u}_\triangleright, \overline{v}_\triangleright)$ respectively. If

$$\begin{aligned} u_r - u_l &\leq D - 2^{n+1}, && \\ u_l - u_i &> 2^n && \text{for all } i < l, \text{ and} \\ u_i - u_r &> 2^n && \text{for all } i > r, \end{aligned}$$

then Duplicator wins the n-round game from configuration $(\overline{u}, \overline{v})$.

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Theorem

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3. Extend result to general games using non-local composition theorem.

Further Results

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For any linear function $f(x) = ax + b$*, the structure* $(\mathbb{R}, <, f)$ *has the 3-variable property.*

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3. "Establish a general model-theoretic characterization of those relational structures that posses the k-variable property for some k."

Immerman and Kozen 97

Perspectives

Two-level composition argument:

Perspectives

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How does this relate to automata-theoretic approaches?